The Langberg-Médard Multiple Unicast Conjecture for 3-Pair Networks *†

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Abstract

The Langberg-Médard multiple unicast conjecture claims that for a strongly reachable k-pair network, there exists a feasible multi-flow with rate $(1, 1, \dots, 1)$. In this paper, we show that the conjecture holds true for $k = 3$.

1 Introduction

A (k, l)-*network* refers to a 4-tuple $\mathcal{N} = (V, A, S, T)$ which consists of a directed acyclic graph $D = (V, A)$, a set $S = \{s_1, s_2, \dots, s_k\}$ of vertices with zero indegree, called *sources* (or *senders*), and a set $T = \{t_1, t_2, \dots, t_l\}$ of vertices with zero outdegree, called *sinks* (or *receivers*). In this paper, we are mainly concerned with (k, k)-networks, typically referred to as k-*pair networks* in the literature and henceforth so in this paper.

Roughly put, the *multiple unicast conjecture*, also known as the *Li-Li multiple unicast conjecture* [12], claims that for any k-pair network, if information can be transmitted from all the senders to their corresponding receivers at rate $(d_1, d_2, \dots, d_k)$ via network coding, then it can be transmitted at the same rate via undirected fractional routing. One of the most challenging problems in the theory of network coding [17], this conjecture has been doggedly resisting a series of attacks [1, 2, 3, 4, 7, 8, 11, 13, 15, 16, 18] and is still open to date.

A (k, l)-network $\mathcal{N}$ is said to be *strongly reachable* if there exists an s_i-t_j directed path P_{s_i,t_j} for all i, j such that $P_{s_1,t_j}, P_{s_2,t_j}, \cdots, P_{s_k,t_j}$ are *arc-disjoint* for any j. Throughout the paper, we will reserve the notations $\mathbf{P}_{t_j}$ and $\mathbf{P}$, where

$$\mathbf{P}_{t_j} := \{P_{s_i,t_j} : i = 1, 2, \dots, k\}, \quad \mathbf{P} := \cup_{j=1}^{l} \mathbf{P}_{t_j}.$$

We emphasize here that the choice of each P_{s_i,t_j} may not be unique, which means that the choice of each $\mathbf{P}_{t_j}$ and that of $\mathbf{P}$ may not be unique.

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The following *Langberg-Médard multiple unicast conjecture* [11], which deals with strongly reachable k-pair networks, is a weaker version of the Li-Li multiple unicast conjecture.

Conjecture 1.1. *For any strongly reachable k-pair network, there exists a feasible undirected fractional multi-flow with rate* $(1, 1, \ldots, 1)$.

The Langberg-Médard multiple unicast conjecture was first proposed in 2009 [11]. In the same paper, the authors constructed a feasible undirected fractional multi-flow with rate $(1/3, 1/3, \ldots, 1/3)$ for strongly reachable k-pair networks. Recently, we have improved $1/3$ to $8/9$ for a generic k in [1] and to $11/12$ for $k = 3, 4$ in [4].

In this paper, we will establish Conjecture 1.1 for $k = 3$. In a nutshell, our approach is based on the ideas and techniques developed in our previous work [1]-[4] and a delicate topological classification of the so-called *minimal* 3-pair networks. Related work on network topology analysis can be found in [10] and [6].

The rest of this paper is organized as follows. In Section 2, we recall some basic notions and facts on the undirected fractional multi-flow. In Section 3, we introduce the notion of regular network which allows us to put Conjecture 1.1 into an equivalent form. Then, in Sections 4 and 5, we analyse the topologies of a minimal $(3, 2)$-network and a minimal 3-pair network, respectively. Finally, our main result is stated and proved in Section 6.

2 Undirected Fractional Multi-Flows

In this section, we consider a strongly reachable k-pair network and adopt the associated notations as introduced in Section 1.

For a directed path or an arc P starting from $u \in V$ and ending at $v \in V$, we say u and v are the *tail* and the *head* of P, denoted by $tail(P)$ and $head(P)$, respectively. For any $s, t \in V$, an *undirected fractional s-t flow* (in simple, an s-t flow) [1] is a function $f : A \to \mathbb{R}$ satisfying the following *flow conservation law:* for any $v \notin \{s, t\}$,

$$excess_f(v) = 0, \tag{1}$$

where

$$excess_f(v) := \sum_{a \in A:\, tail(a)=v} f(a) - \sum_{a \in A:\, head(a)=v} f(a). \tag{2}$$

It is easy to see that for any s-t flow f, $excess_f(s) = -excess_f(t)$, which is called the *rate (or value)* of f. We say f is *feasible* if $|f(a)| \le 1$ for all $a \in A$.

Given a k-pair network $\mathcal{N} = (V, A, S, T)$, for any $i = 1, 2, \ldots, k$, let f_i be an s_i-t_i flow, where $s_i \in S$ and $t_i \in T$. We will refer to $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ as an *undirected fractional multi-flow* (in simple, a multi-flow) with *rate* $(d_1, d_2, \ldots, d_k)$, where d_i is the rate of f_i. $\mathcal{F}$ is said to be *feasible* if

$$\sum_{1 \le i \le k} |f_i(a)| \le 1$$

for all $a \in A$.

[1] The undirected fractional flow/multi-flow defined for directed graph in this paper, which can be negative, is equivalent to that defined in [14] for undirected graphs, which has to be non-negative.

For a strongly reachable network $\mathcal{N}$, we can define the *linear* multi-flow as follows. Given a choice $\mathbf{P}$ of $\mathcal{N}$, for each $P_{s_i,t_j} \in \mathbf{P}$, we first define an s_i-t_j flow $f_{i,j}$ as

$$f_{i,j}(a) = \begin{cases} 1, & a \text{ belongs to } P_{s_i,t_j}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, a multi-flow $\mathcal{F} = \{f_1, f_2, \ldots, f_k\}$ is said to be *linear* (by default, with respect to $\mathbf{P}$) if for each feasible l, there exist $c_{i,j}^{(l)} \in \mathbb{R}$, $1 \leq i, j \leq k$, such that

$$f_l = \sum_{i,j=1}^{k} c_{i,j}^{(l)} f_{i,j}, \tag{3}$$

in which case $\mathcal{F}$ can be equivalently represented by its *matrix form*

$$\mathcal{C} = \left((c_{i,j}^{(1)}), (c_{i,j}^{(2)}), \ldots, (c_{i,j}^{(k)})\right);$$

otherwise, it is called *non-linear*.

The following theorem has been established in [4].

Theorem 2.1. *A linear multi-flow $\mathcal{F}$ has rate $(1, 1, \ldots, 1)$ if and only if all $c_{i,j}^{(l)}$ in (3) satisfy*

$$\sum_{j=1}^{k} c_{i,j}^{(l)} = 0, \text{ for all } i \neq l, \quad \sum_{i=1}^{k} c_{i,j}^{(l)} = 0, \text{ for all } j \neq l, \quad \sum_{i=1}^{k}\sum_{j=1}^{k} c_{i,j}^{(l)} = 1, \text{ for all } l. \tag{4}$$

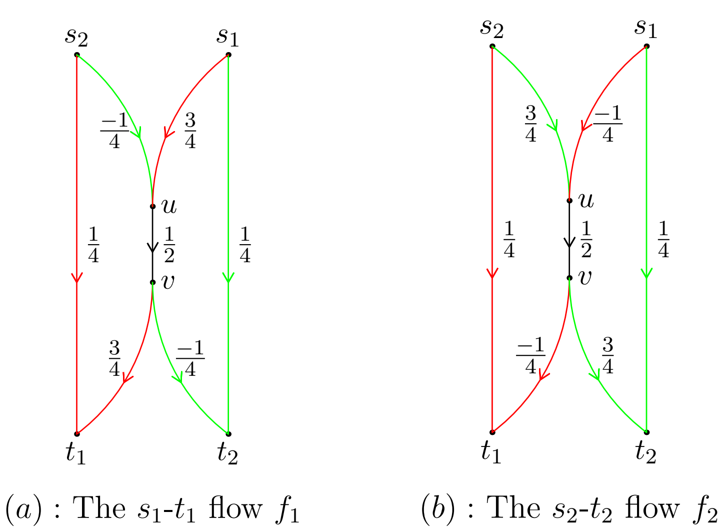

Figure 1: A linear multi-flow.

Example 2.2. Consider the butterfly network depicted in Fig. 1, which is a strongly reachable 2-pair network with an unique choice of $\mathbf{P} = \{P_{s_i,t_j} : 1 \leq i, j \leq 2\}$. It is easy to check that the multi-flow $\mathcal{F} = (f_1, f_2)$ given in Fig. 1 is linear with rate $(1, 1)$ and the corresponding matrix form is

$$\left(\left(\begin{array}{cc} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{-1}{4} \end{array} \right), \left(\begin{array}{cc} \frac{-1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{array} \right) \right),$$

i.e., $f_1 = \frac{3}{4} f_{1,1} + \frac{1}{4} f_{1,2} - \frac{1}{4} f_{2,2} + \frac{1}{4} f_{2,1}$ and $f_2 = \frac{3}{4} f_{2,2} + \frac{1}{4} f_{2,1} - \frac{1}{4} f_{1,1} + \frac{1}{4} f_{1,2}$. Noting that

$$|f_1(a)| + |f_2(a)| = \begin{cases} \frac{1}{2}, & a \in \{[s_1, t_2], [s_2, t_1]\}; \\ 1, & \text{otherwise}, \end{cases}$$

we see that $\mathcal{F}$ is feasible. On the other hand, it can be verified that the multi-flow $\mathcal{F} = \{f_1, f_2\}$ given in Fig. 2 also has rate $(1, 1)$ but is non-linear.

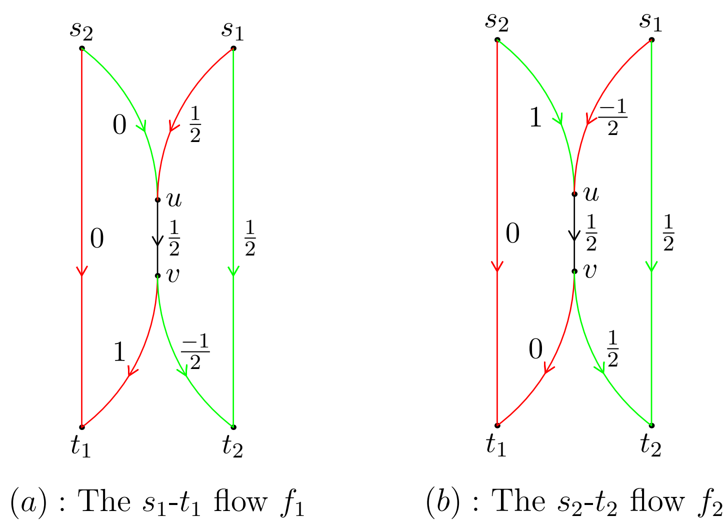

Figure 2: A non-linear multi-flow.

3 Regular (k, l)-Networks

We start with the definition of regular network.

Definition 3.1. [Regular (k, l)-network] A (k, l)-network $\mathcal{N} = (V, A, S, T)$ is said to be *regular* if it is strongly reachable and for each $v \in V \setminus T$, either the indegree $deg^+(v) = 1$ or the outdegree $deg^-(v) = 1$.

In this paper, we will establish Conjecture 1.1 for $k = 3$ by proving the following equivalent conjecture. Note that the only difference between the two conjectures is that "strongly reachable" in Conjecture 1.1 is replaced by "regular" in Conjecture 3.2.

Conjecture 3.2. *For any regular k-pair network, there exists a feasible undirected fractional multi-flow with rate* $(1, 1, \ldots, 1)$.

To see the equivalence between the two conjectures, we first note that Conjecture 1.1 trivially implies Conjecture 3.2. To prove the other direction, for a strongly reachable k-pair network $\mathcal{N}$, we perform the following operations: (1) for each vertex w with indegree $deg^+(w) = m > 1$ and outdegree $deg^-(w) = n > 1$, we replace it by a directed bipartite graph $K_{m,n}$ with vertex set $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$ and arc set $\{[u_i, v_j] : 1 \leq i \leq m; 1 \leq j \leq n\}$, and moreover, replace each of its incoming arc $[w'_i, w]$ by $[w'_i, u_i]$, $i = 1, 2, \ldots, m$, and each of its outgoing arc $[w, w''_j]$ by $[v_j, w''_j]$, $j = 1, 2, \ldots, n$; (2) for each source s_i with outdegree $deg^-(s_i) = n > 1$, we replace s_i by an arc $[s_i, s'_i]$ and replace each of its outgoing arc $[s_i, v_j]$ by $[s'_i, v_j]$, $j = 1, 2, \ldots, n$. It can then be readily verified that the newly obtained network $\mathcal{N}'$ is regular and each feasible undirected fractional multi-flow of $\mathcal{N}'$ can be mapped to one in $\mathcal{N}$ with the same rate. Hence, Conjecture 3.2 implies Conjecture 1.1.

Henceforth, we assume $\mathcal{N} = (V, A, S, T)$ is a regular (k, l)-network. To avoid cumbersome wording, we refer to a path from $\mathbf{P}_{t_j}$ as a $\mathbf{P}_{t_j}$-path, and moreover, an arc on a $\mathbf{P}_{t_j}$-path as a $\mathbf{P}_{t_j}$-arc. Here, we note that a $\mathbf{P}_{t_j}$-arc can be simultaneously a $\mathbf{P}_{t_{j'}}$-arc for some $j' \neq j$; if such j' does not exist, the arc is said to be a *pure* $\mathbf{P}_{t_j}$-arc.

For ease of presentation, we will henceforth follow [9] and "draw" (or in geometrical terms, "embed") $\mathcal{N}$ in $\mathbb{R}^3$, and as such, $\mathcal{N}$ will be treated as *a set of points* in $\mathbb{R}^3$. More specifically, we identify each vertex $v \in V$ as a point in $\mathbb{R}^3$, still denoted by v, and each arc $[u, v] \in A$ as a *curve* [2] $[u, v] \subset \mathbb{R}^3$ such that for any two arcs $[u_1, v_1], [u_2, v_2] \in A$, the corresponding curves satisfy $[u_1, v_1] \cap [u_2, v_2] = \{u_1, v_1\} \cap \{u_2, v_2\}$.

With this viewpoint, it is possible to state the following theorem (and all the results in later sections) using set-theoretical and topological notions.

Theorem 3.3. *For a regular (k, l)-network $\mathcal{N}$, the following statements hold.*

1) *For any u_1-v_1 path P_1 and u_2-v_2 path P_2, $P_1^\circ \cap P_2^\circ$ contains no isolated point, where $P_i^\circ = P_i \setminus \{u_i, v_i\}$ for $i = 1, 2$.*

2) *For any s_{i_1}-t_{j_1} path P_1 and s_{i_2}-t_{j_2} path P_2, if $j_1 \neq j_2$, then $P_1 \cap P_2$ contains no isolated point; in particular, for any $P_1 \in \mathbf{P}_{t_{j_1}}$ and $P_2 \in \mathbf{P}_{t_{j_2}}$, if $j_1 \neq j_2$, then $P_1 \cap P_2$ contains no isolated point.*

3) *For any fixed j, $\cap_{P \in \mathbf{P}_{t_j}} P = \{t_j\}$.*

Proof. By Definition 3.1, there is no isolated point $v \in P_1^\circ \cap P_2^\circ$ since otherwise v would be a vertex of $\mathcal{N}$ whose indegree and outdegree are both at least 2. Hence 1) holds. By the definition, each source of a regular (k, l)-network has indegree zero and outdegree 1, and hence 2) holds by applying 1). Noticing the arc-disjointness of all $\mathbf{P}_{t_j}$-paths for a fixed j and by applying 1), we conclude that 3) holds. □

Remark 3.4. Statements 2, 3) of Theorem 3.3, which do not hold true for a strongly reachable (k, l)-network that is not regular, will play an important role in topologically categorizing regular $(3, 2)$-networks in the next section.

Definition 3.5. [Semi-Cycle [6]] Given a choice of $\mathbf{P}_{t_j}$, a subgraph C of $\mathcal{N}$ is said to be a $\mathbf{P}_{t_j}$-*semi-cycle* if it becomes a directed cycle after reversing the direction of each $\mathbf{P}_{t_j}$-arc (not necessarily pure).

[2] A curve $[u, v]$ is the image of the unit interval $[0, 1]$ under a homeomorphism such that 0 is mapped to $u \in \mathbb{R}^3$ and 1 to $v \in \mathbb{R}^3$.

A $\mathbf{P}_{t_j}$-semi-cycle C is said to be of *order* n if $\widehat{C} := C \cap (\cup_{P\in\mathbf{P}_{t_j}} P)$ has n connected components $[u_i, v_i]$, $i = 1, 2, \ldots, n$. It is easy to see that $\widehat{C}$ has no isolated point. Hence, all $[u_i, v_i]$ are subpaths of $\mathbf{P}_{t_j}$-paths and we will call each u_i (*resp.* v_i) a *tail* (*resp.* *head*) of C. For each $\mathbf{P}_{t_j}$-semi-cycle C, clearly, $(C \cup_{P\in\mathbf{P}_{t_j}} P \setminus \widehat{C}) \cup_{i=1}^{n} \{u_i, v_i\}$ provides an alternative choice of $\mathbf{P}_{t_j}$. For this reason we call $(C \setminus \widehat{C}) \cup_{i=1}^{n} \{u_i, v_i\}$ a $\mathbf{P}_{t_j}$-*crossing* of order n associated to C. Note that for this alternative choice of $\mathbf{P}_{t_j}$, C is still a $\mathbf{P}_{t_j}$-semi-cycle while $\widehat{C}$ is the associated $\mathbf{P}_{t_j}$-crossing. Consequently, a $\mathbf{P}_{t_j}$-semi-cycle C can be written as the union of two associated $\mathbf{P}_{t_j}$-crossings, each corresponding to a choice of $\mathbf{P}_{t_j}$. More formally, we will often in the following write $C = \widehat{C_1} \cup \widehat{C_2}$, where $\widehat{C_1}$ (*resp.* $\widehat{C_2}$) is a $\mathbf{P}_{t_j}$-crossing if $\widehat{C_2} \subset \cup_{P\in\mathbf{P}_{t_j}} P$ (*resp.* $\widehat{C_1} \subset \cup_{P\in\mathbf{P}_{t_j}} P$); see Fig. 3 for an illustrative example.

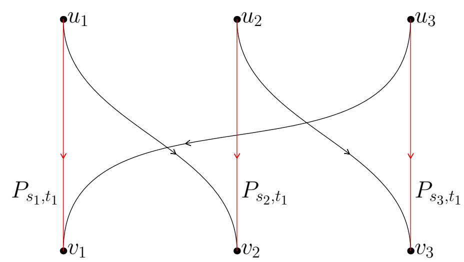

Figure 3: A $\mathbf{P}_{t_1}$-semi-cycle C of order 3 with heads $\{v_1, v_2, v_3\}$ and tails $\{u_1, u_2, u_3\}$. $C = \widehat{C_1} \cup \widehat{C_2}$ where $\widehat{C_1} = [u_1, v_1] \cup [u_2, v_2] \cup [u_3, v_3]$ and $\widehat{C_2} = [u_1, v_2] \cup [u_2, v_3] \cup [u_3, v_1]$. If $\widehat{C_1} \subset \cup_{P\in\mathbf{P}_{t_1}} P$ (as shown in the figure) then $\widehat{C_2}$ is a $\mathbf{P}_{t_1}$-crossing and if $\widehat{C_2} \subset \cup_{P\in\mathbf{P}_{t_1}} P$ then $\widehat{C_1}$ is a $\mathbf{P}_{t_1}$-crossing.

Lemma 3.6. *A regular (k, l)-network $\mathcal{N}$ has no $\mathbf{P}_{t_j}$-semi-cycle/crossing if and only if the choice of $\mathbf{P}_{t_j}$ is unique.*

Proof. The "if " part is trivial. For the other direction, we suppose there exist two choices of $\mathbf{P}_{t_j}$, say, $\mathbf{P}_{t_j}^{(1)}$ and $\mathbf{P}_{t_j}^{(2)}$. Then,

$$\mathcal{N}'_{t_j} := \bigcup_{P_1\in\mathbf{P}_{t_j}^{(1)}, P_2\in\mathbf{P}_{t_j}^{(2)}} (P_1 \cup P_2) \setminus (P_1 \cap P_2)$$

is not empty. Now, we reverse the direction of each $\mathbf{P}_{t_j}^{(1)}$-arc in $\mathcal{N}'_{t_j}$ to obtain $\mathcal{N}''_{t_j}$. Then, noticing that $\cap_{P\in\mathbf{P}_{t_j}} P = \{t_j\}$ for each choice of $\mathbf{P}_{t_j}$, we infer that for any $v \neq t_j$, $deg^-(v) = deg^+(v) = 1$ and $deg^-(t_j) = deg^+(t_j)$. Hence, $\mathcal{N}''_{t_j}$ is an Eulerian directed graph and composed of arc-disjoint directed cycles, each corresponding to a $\mathbf{P}_{t_j}$-semi-cycle of $\mathcal{N}$ by definition. □

Remark 3.7. According to the proof of Lemma 3.6, we can have that: For any $\mathbf{P}_{t_j}$-arc, either it is a $\mathbf{P}_{t_j}$-arc for *all* choice of $\mathbf{P}_{t_j}$ or it belongs to a $\mathbf{P}_{t_j}$-semi-cycle.

Definition 3.8. [Minimal (k, l)-Network] A (k, l)-network $\mathcal{N}$ is said to be *minimal* if it is regular yet ceases to be regular upon removal of any of its arcs.

In the following two sections, we will investigate the topological properties of minimal $(3, 2)$-networks and minimal 3-pair networks.

4 Minimal $(3,2)$-Networks

In this section, we aim to derive all the possible topologies of a minimal $(3,2)$-network. To that end, we first consider a regular $(3,2)$-network $\mathcal{N}$ and obtain some lemmas, which are also useful in the next section. For notational convenience, for any choice of $\mathbf{P}_{t_1}$ and $\mathbf{P}_{t_2}$ of a regular $(3,2)$-network $\mathcal{N}$, we rewrite $P_{s_i,t_1} \in \mathbf{P}_{t_1}$ and $P_{s_i,t_2} \in \mathbf{P}_{t_2}$ as r_i and g_i, $i = 1,2,3$, respectively, and rewrite $\mathbf{P}_{t_1}$ and $\mathbf{P}_{t_2}$ as r and g, respectively.

4.1 Useful Lemmas

In this subsection, we consider a regular $(3,2)$-network $\mathcal{N}$, which is not necessarily minimal.

Lemma 4.1. *A regular $(3,2)$-network $\mathcal{N}$ is minimal if and only if $\mathcal{N} = \cup_{p \in r \cup g} p$ and has no semi-cycle (i.e., r-semi-cycle or g-semi-cycle).*

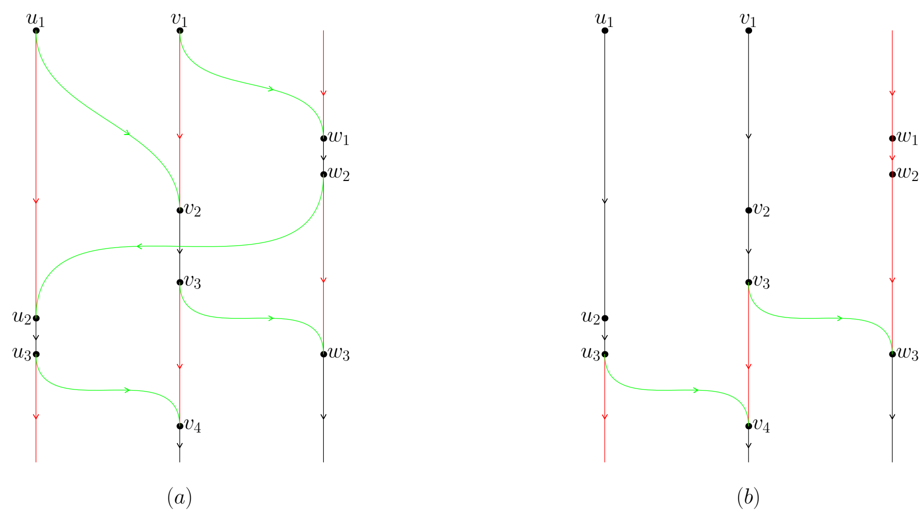

Figure 4: Illustrations for the proof of Lemma 4.1, where pure g-arcs and pure r-arcs are colored green and red, respectively. (a) illustrates a g-semi-cycle C consisting of $[u_1, v_2]$, $[v_1, w_1, w_2, u_2]$ (each a subpath of some g-path) and $[u_1, u_2]$, $[v_1, v_2]$ (each a subpath of some r-path). Note that C is not an r-semi-cycle. As shown in (b), after removing all pure g-arcs in C, one can find an alternative g.

Proof. The "if" part immediately follows from the fact that both r and g are unique due to Lemma 3.6. For the other direction, suppose r and g be a choice of $\mathbf{P}_{t_1}$ and $\mathbf{P}_{t_2}$, clearly we have $\mathcal{N} = \cup_{p \in r \cup g} p$ by the minimality of $\mathcal{N}$. Now, we assume, without loss of generality, there is a g-semi-cycle C. Then, C has at least one *pure* g-arc, since otherwise all arcs of C would be r-arcs, which is impossible by 3) of Theorem 3.3. Removing all the *pure* g-arcs within C, we will obtain a regular $(3,2)$-network (see Fig. 4 for an illustrative example), contradicting the minimality of $\mathcal{N}$. □

Given a regular $(3,2)$-network $\mathcal{N}$ and a choice of r and g, for any $r_i \in r$, let $r_i|_g$ denote the intersection of r_i with g-paths, or more precisely,

$$r_i|_g := r_i \cap (\cup_{j=1}^{3} g_j). \tag{5}$$

Let $\ell^g(r_i)$ be the number of connected components of $r_i|_g$ and $r_i^g(1), r_i^g(2), \ldots, r_i^g(\ell^g(r_i))$ be all the connected components listed from upstream to downstream along the direction of r_i. According to Theorem 3.3, for each feasible l, $r_i^g(l)$ is a subpath of r_i (rather than an isolated point). Denote by $r_i^g(l, l+1)$ the subpath of r_i from the head of $r_i^g(l)$ to the tail of $r_i^g(l+1)$. Note that with r and g swapped, notations like $g_i|_r$ and $g_i^r(l)$ can be defined in a parallel fashion. As there are only two groups of paths under consideration, we may in the following omit the superscripts and write $r_i^g(l), \ell^g(r_i), g_i^r(l), \ell^r(g_i)$ as $r_i(l), \ell(r_i), g_i(l), \ell(g_i)$, respectively.

Note that by the regularity of $\mathcal{N}$, each source s_i has indegree zero and outdegree 1, and hence for any choice of r and g, we have $r_i(1) = g_i(1)$, for all $i = 1, 2, 3$. Given a choice of r and g, a semi-cycle/crossing is said to be *uppermost* if each tail of the semi-cycle/crossing is a head of $r_i(1) = g_i(1)$ for some feasible i. We have the following lemmas.

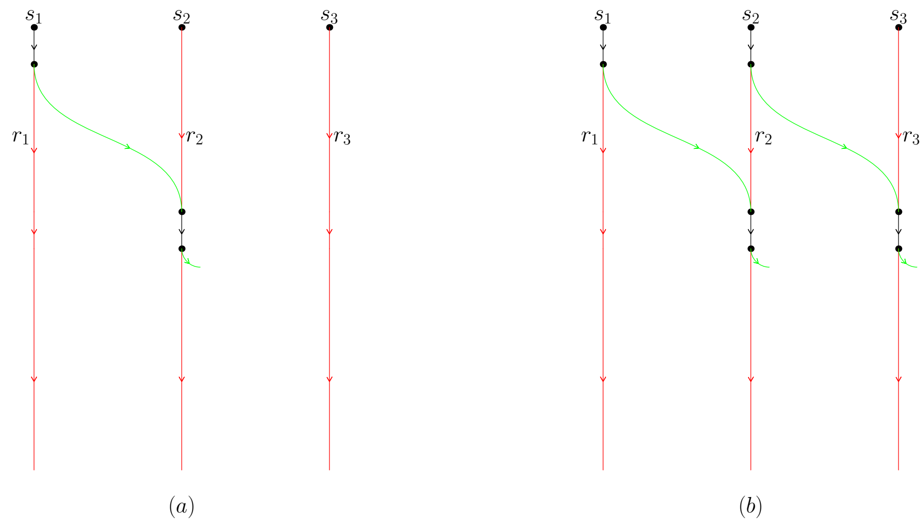

Figure 5: Illustrations for the proof of Lemma 4.2.

Lemma 4.2. *Given a choice of r and g of $\mathcal{N}$, if there exists no uppermost r-semi-cycle, then there exists $g_i \in g$ such that $\ell(g_i) = 1$; in particular, if $\mathcal{N}$ is minimal, then there exists $g_i \in g$ such that $\ell(g_i) = 1$.*

Proof. First of all, we consider the path g_1. If $\ell(g_1) = 1$, then there is nothing to prove. Hence, we suppose in the following that $\ell(g_1) > 1$. If $g_1(2) \subseteq r_1$, then $g_1(1,2)$ forms an r-crossing such that its tail is the head of $r_1(1) = g_1(1)$. Hence, we assume in the following that $g_1(2) \subseteq r_2$ (see Fig. 5(a)).

Now, we consider the path g_2. As before, we suppose in the following that $\ell(g_2) > 1$. By the same arguments as above, we have $g_2(2) \nsubseteq r_2$. Moreover, it holds that $g_2(2) \nsubseteq r_1$ since otherwise $g_1(1,2)$ and $g_2(1,2)$ form an r-crossing of order 2 such that its two tails are the heads of $r_1(1) = g_1(1)$ and $r_2(1) = g_2(1)$, which implies that $g_2(2) \subseteq r_3$ (see Fig. 5(b)).

Finally, consider the path g_3 and suppose, by way of contradiction, that $\ell(g_3) > 1$. Then, similarly as above, we have $g_3(2) \nsubseteq r_3$ and $g_3(2) \nsubseteq r_2$. Moreover, we have $g_3(2) \nsubseteq r_1$ since otherwise $g_1(1,2)$, $g_2(1,2)$ and $g_3(1,2)$ form an r-crossing of order 3 such that its three tails are the heads of $r_1(1) = g_1(1)$, $r_2(1) = g_2(1)$ and $r_3(1) = g_3(1)$. Hence, we obtain a contradiction to the existence of $g_3(2)$ and thus $\ell(g_3) = 1$, as desired. $\square$

By swapping r and g of Lemma 4.2, we have that if there exists no uppermost g-semi-cycle, then there exists $r_j \in r$ such that $\ell(r_j) = 1$. In fact, we can further have the following lemma.

Lemma 4.3. *Given a choice of r and g of $\mathcal{N}$, if there exists no uppermost r-semi-cycle, then for any j such that $\ell(r_j) = 1$, there exists $i \neq j$ such that $\ell(g_i) = 1$; in particular, if $\mathcal{N}$ is minimal, then for any j such that $\ell(r_j) = 1$, there exists $i \neq j$ such that $\ell(g_i) = 1$.*

Proof. Without loss of generality assume $\ell(r_1) = 1$. By Lemma 4.2, there exists $g_i \in g$ such that $\ell(g_i) = 1$. If $i \neq 1$, then the lemma is true. So, we suppose in the following $\ell(g_1) = 1$.

We first consider the path g_2. If $\ell(g_2) = 1$, then the lemma is true. Hence, we suppose in the following $\ell(g_2) > 1$. Clearly, $g_2(2) \not\subseteq r_2$ since otherwise $g_2(1, 2)$ forms an uppermost r-crossing; and $g_2(2) \not\subseteq r_1$ since $\ell(r_1) = 1$. Hence, $g_2(2) \subseteq r_3$.

Now, consider the path g_3. If $\ell(g_3) = 1$, then we have done. Hence, we suppose in the following $\ell(g_3) > 1$. Clearly, $g_3(2) \not\subseteq r_3$ since otherwise $g_3(1, 2)$ forms an uppermost r-crossing; and $g_3(2) \not\subseteq r_2$ since otherwise $g_2(1, 2)$ and $g_3(1, 2)$ form an uppermost r-crossing. It then follows that $g_3(2) \subseteq r_1$, which however contradicts $\ell(r_1) = 1$. $\square$

Clearly, by swapping r with g of Lemma 4.3, we have that if there exists no uppermost semi-cycle, then for any i such that $\ell(g_i) = 1$, there exists $j \neq i$ such that $\ell(r_j) = 1$.

Definition 4.4. [pseudo-minimal $(3, 2)$-network] A regular $(3, 2)$-network $\mathcal{N}$ is said to be *pseudo-minimal* if there exists a choice of r and g such that $\mathcal{N} = \cup_{p \in r \cup g} p$ and $i \neq j$ such that $\ell(g_i) = \ell(r_j) = 1$.

According to Lemma 4.2 and Lemma 4.3, if $\mathcal{N} = \cup_{p \in r \cup g} p$ has no uppermost semi-cycle, then it is pseudo-minimal; in particular, a minimal $(3, 2)$-network $\mathcal{N}$ is pseudo-minimal. This fact will play a key role in our treatment.

4.2 Topologies of Minimal $(3, 2)$-network

In this subsection, we will derive all the possible topologies of a minimal $(3, 2)$-network $\mathcal{N}$. We say $\mathcal{N} = \cup_{p \in r \cup g} p$ is *non-degenerated* if there exists an unique choices of $i \neq j$ such that $\ell(g_i) = \ell(r_j) = 1$, and *degenerated* otherwise. We will first deal with the degenerated case and then the non-degenerated case. The following simple observation will be frequently used.

Lemma 4.5. *For any feasible l, if $r_i(l) \subseteq g_j$, then $r_i(l + 1) \not\subseteq g_j$; similarly, if $g_i(l) \subseteq r_j$, then $g_i(l + 1) \not\subseteq r_j$.*

Proof. If $r_i(l) \subseteq g_j$ and $r_i(l + 1) \subseteq g_j$, then $r_i(l, l + 1)$ forms a g-crossing, contradicting the minimality of $\mathcal{N}$. $\square$

Theorem 4.6. *A degenerated $\mathcal{N}$ has three possible topologies as in Fig. 6.*

Proof. We will deal with the following two cases:

1) There exists i such that $\ell(r_i) = \ell(g_i) = 1$. In this case, by Lemma 4.3, we have the following subcases:

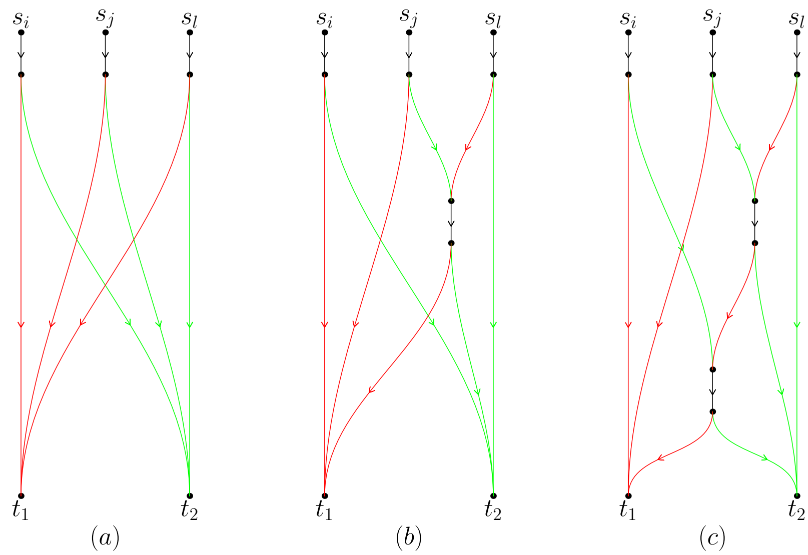

Figure 6: Possible topologies of a degenerated $\mathcal{N}$

1.1) There exists $j \neq i$ such that $\ell(g_j) = 1$ and $\ell(r_j) = 1$. In this case, it is easy to see that there exists l distinct from both i and j such that $\ell(r_l) = \ell(g_l) = 1$, as shown in (a) of Fig. 6.

1.2) There exist $j \neq i$ and $l \neq i$ such that $\ell(r_j) = 1$ and $\ell(g_l) = 1$. In this case, if $\ell(g_j) = 1$, we have Case 1.1); otherwise, we have $g_j(2) = r_l(2)$, as shown in (b) of Fig. 6.

2) For all $i = 1, 2, 3$, $\ell(r_i) = \ell(g_i) = 1$ does not hold. In this case, there exist distinct i, j, l such that $\ell(r_i) = \ell(r_j) = \ell(g_l) = 1$. We consider r_l and have either "$r_l(2) = g_j(2)$ and $r_l(3) = g_i(2)$", as shown in (c) of Fig. 6, or "$r_l(2) = g_i(2)$ and $r_l(3) = g_j(2)$", as shown in (c) of Fig. 6 with i, j swapped.

□

Theorem 4.7. *A non-degenerated $\mathcal{N}$ has five possible topologies as in Fig. 7.*

Proof. For a non-degenerated $\mathcal{N}$, we suppose that $\ell(r_i) = \ell(g_l) = 1$ and start our argument by considering $g_i(2)$ and $g_j(2)$. By Lemma 4.5, we have the following two cases:

1) $g_i(2) \subseteq r_j$ and $g_j(2) \subseteq r_l$. In this case, by Lemma 4.5, we infer that $r_j(2) \not\subseteq g_j$ and hence $r_j(2) \subseteq g_i$, which implies that $g_i(2) = r_j(2)$ (due to the acyclicity of $\mathcal{N}$). We then further consider the following two subcases:

1.1) $\ell(g_i) = 2$.

1.2) $\ell(g_i) > 2$.

In Case 1.1), it is easy to see that $g_j(2) = r_l(2)$, which leads to the following two subcases:

1.1.1) $\ell(g_j) = 2$. In this case, $\mathcal{N}$ has the topology as in (a) of Fig. 7.

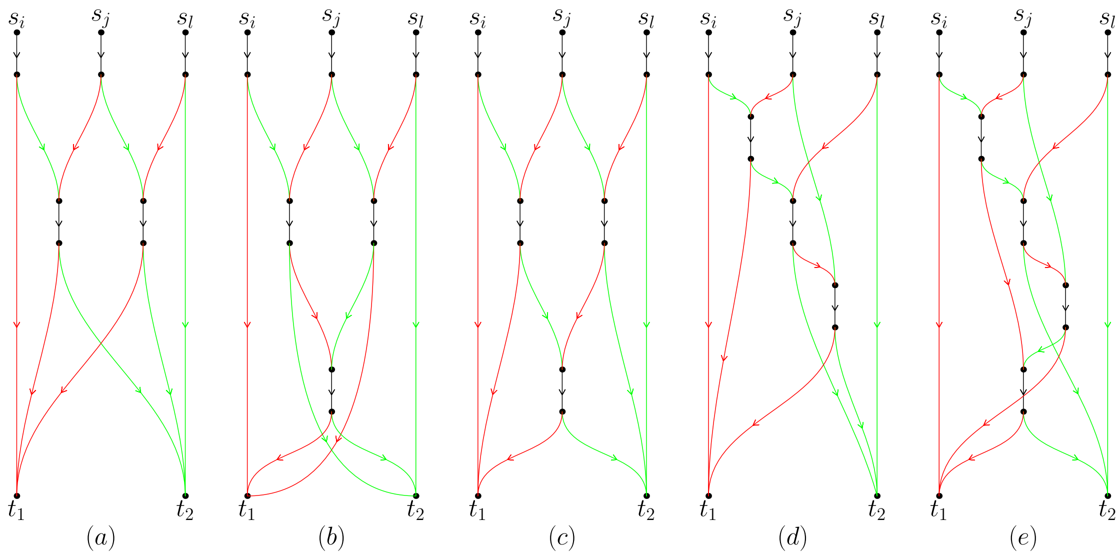

Figure 7: Possible topologies of a non-degenerated $\mathcal{N}$

1.1.2) $\ell(g_j) \geq 3$. In this case, $g_j(3) \subseteq r_j$. By Lemma 4.5 and the acylicity of the network, we have $g_j(3) = r_j(3)$. Moreover, it holds true that $\ell(g_j) = 3$ since otherwise if $g_j(4) \subseteq r_l$, again by Lemma 4.5 and the acyclicity of the network, we have $g_j(4) = r_l(3)$ and hence $r_l(2), r_l(3) \subseteq g_j$, which contradicts Lemma 4.5. Hence, in this case, $\mathcal{N}$ is topologically equivalent to (b) of Fig. 7.

In Case 1.2), since $g_i(2) \subseteq r_j$, we have $g_i(3) \subseteq r_l$. Since $g_j(2), g_i(3) \subseteq r_l$, we will deal with the following two subcases:

1.2.1) $g_j(2) = r_l(2)$ and $g_i(3) = r_l(3)$. In this case, if $\ell(g_j) \geq 3$, then by Lemma 4.5, $g_j(3) \subseteq r_j$, which however would imply $g_j(2,3)$ and $g_i(2,3)$ form an r-crossing. Hence, we have $\ell(g_j) = 2$. Now, if $\ell(g_i) \geq 4$, then by Lemma 4.5, $g_i(4) \subseteq r_j$, which further implies $g_i(4) = r_j(3)$. Hence, $r_j(2), r_j(3) \subseteq g_i$, which contradicts Lemma 4.5. Hence $\ell(g_j) = 2, \ell(g_i) = 3$, and $\mathcal{N}$ has the topology as in (c) of Fig. 7.

1.2.2) $g_i(3) = r_l(2)$ and $g_j(2) = r_l(3)$. In this case, if $\ell(g_i) \geq 4$, then $g_i(4) \subseteq r_j$ and hence $g_i(3,4)$ and $g_j(1,2)$ form an r-crossing, which is a contradiction. Thus, $\ell(g_i) = 3$ and then we will consider the following two subcases:

1.2.2.1) $\ell(g_j) = 2$. In this case, we conclude that $\mathcal{N}$ has the topology as in (d) of Fig. 7.

1.2.2.2) $\ell(g_j) \geq 3$. In this case, by Lemma 4.5, we have $g_j(3) \subseteq r_j$, which further implies $g_j(3) = r_j(3)$. Now, if $\ell(g_j) \geq 4$, then by Lemma 4.5, $g_j(4) \subseteq r_l$, which further implies $g_j(4) = r_l(4)$ and hence $r_l(3), r_l(4) \subseteq g_j$, which contradicts Lemma 4.5. Hence $\ell(g_j) = 3$ and we conclude that $\mathcal{N}$ has the topology as in (e) of Fig. 7.

2) $g_i(2) \subseteq r_l$ and $g_j(2) \subseteq r_l$. In this case, via a relabelling if necessary, we can assume $g_j(2) = r_l(2)$ and $g_i(2) = r_l(3)$. By Lemma 4.5, we have $g_i(3) \subseteq r_j$ and $g_j(3) \subseteq r_j$. Then, we will consider the following two subcases:

2.1) $g_j(3) = r_j(2)$;

2.2) $g_i(3) = r_j(2)$.

It is easy to see that 2.1) is impossible since otherwise $r_j(1), r_j(2) \subseteq g_j$, which contradicts Lemma 4.5. Hence, we have $r_j(2) \subseteq g_i$ and $r_l(2) \subseteq g_j$. By switching the labels of sinks and relabeling sources s_i, s_l as s_l, s_i, respectively, we will reach Case 1), which has been dealt with before.

□

The following corollary can be obtained by inspecting (a)-(e) of Fig. 7 and will be used later. For two vertices $u, v \in V$, denote by $u \leq v$ if $u = v$ or there exists an u-v directed path; and for two paths p_1, p_2, denote by $p_1 \leq p_2$ if $tail(p_1) \leq tail(p_2)$.

Corollary 4.8. *For a non-degenerated $\mathcal{N}$ with $\ell(r_i) = \ell(g_l) = 1$, we have:*

1) $\emptyset \neq g_i \cap r_j = r_j(2)$ *and* $\emptyset \neq r_l \cap g_j = g_j(2)$.

2) *Either* $r_l \cap g_i = \emptyset$ *(i.e.,* $\ell(r_l) = \ell(g_i) = 2$*) or* $r_l \cap g_i \neq \emptyset$ *(i.e.,* $\ell(r_l) = \ell(g_i) = 3$*).*

3) *If* $r_l \cap g_i \neq \emptyset$*, then either*

$$r_l \cap g_j \leq r_l \cap g_i \text{ and } r_j \cap g_i \leq r_l \cap g_i, \tag{6}$$

or

$$r_l \cap g_j \leq r_l \cap g_i \leq r_j \cap g_i \tag{7}$$

or

$$r_j \cap g_i \leq r_l \cap g_i \leq r_l \cap g_j. \tag{8}$$

For example, (a)-(b) of Fig. 7 satisfy $r_l \cap g_i = \emptyset$ and (c)-(e) of Fig. 7 satisfy $r_l \cap g_i \neq \emptyset$ (more specifically, (c) satisfies (6) while (d)-(e) satisfy (7)). Clearly, for any non-degenerated $\mathcal{N}$ with $\ell(r_i) = \ell(g_l) = 1$, by swapping the two sinks t_1, t_2 (i.e., swapping r, g) and swapping the two sources s_i, s_l (which keeps the topology of $\mathcal{N}$ unchanged), the property $\ell(r_i) = \ell(g_l) = 1$ still holds. However, this transformation maintains the inequality (6) but swaps the inequalities (7) and (8). Note that the two networks satisfying (8) are not drawn in Fig. 7 since they have the same topology as in (d) and (e).

5 Minimal 3-pair Networks

Throughout this section, we consider a minimal 3-pair network $\mathcal{N}$. We say that $\mathcal{N}$ is *stable* if the choice of $\mathbf{P}$ is unique, and *unstable* otherwise. For any feasible $i \neq j$ and any choice of $\mathbf{P}_{t_i}$ and $\mathbf{P}_{t_j}$, let $\mathcal{N}_{t_i,t_j} := \cup_{P \in \mathbf{P}_{t_i} \cup \mathbf{P}_{t_j}} P$ be the regular $(3,2)$-network induced by $\mathbf{P}_{t_i}$ and $\mathbf{P}_{t_j}$. The following fact will be frequently used: for each minimal 3-pair network $\mathcal{N}$ and each feasible $i \neq j$, there exists a choice of $\mathbf{P}_{t_i}$ and $\mathbf{P}_{t_j}$ such that $\mathcal{N}_{t_i,t_j}$ is a minimal $(3,2)$-network. Note that according to Lemma 3.6, $\mathcal{N}$ is unstable if and only if it contains at least one $\mathbf{P}_{t_l}$-semi-cycle C, which is not necessarily a $\mathbf{P}_{t_l}$-semi-cycle in $\mathcal{N}_{t_i,t_l}$ or a $\mathbf{P}_{t_l}$-semi-cycle in $\mathcal{N}_{t_j,t_l}$, as illustrated in Fig. 8. Throughout this section, we assume that $i, j, l \in \{1, 2, 3\}$ are distinct.

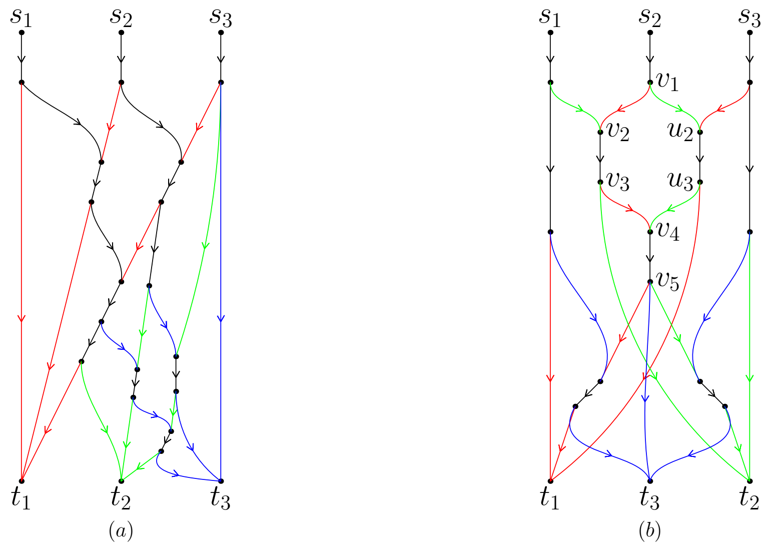

Figure 8: Two minimal 3-pair networks, where (a) is stable and (b) is unstable. In (b), $P_{s_2,t_3} \in \mathbf{P}$ can be chosen as either $[s_2, v_1, v_2, v_3, v_4, v_5, t_3]$ or $[s_2, v_1, u_2, u_3, v_4, v_5, t_3]$, and there is a $\mathbf{P}_{t_3}$-semi-cycle $C = \widehat{C_1} \cup \widehat{C_2}$ of order 1 such that $\widehat{C_1} = [v_1, v_2, v_3, v_4]$ and $\widehat{C_2} = [v_1, u_2, u_3, v_4]$. If P_{s_2,t_3} is chosen such that $\widehat{C_1} \subset P_{s_2,t_3}$ (*resp.* $\widehat{C_2} \subset P_{s_2,t_3}$), then $\widehat{C_2}$ (*resp.* $\widehat{C_2}$) is a $\mathbf{P}_{t_3}$-crossing in $\mathcal{N}_{t_2,t_3}$ (*resp.* $\mathcal{N}_{t_1,t_3}$) but not a $\mathbf{P}_{t_3}$-crossing in $\mathcal{N}_{t_1,t_3}$ (*resp.* $\mathcal{N}_{t_2,t_3}$).

5.1 The Unstable 3-pair Network

We need the following several lemmas on an unstable 3-pair network $\mathcal{N}$.

Lemma 5.1. *Any $\mathbf{P}_{t_l}$-semi-cycle C is covered by $\mathcal{N}_{t_i,t_j}$, i.e., $C \subset \cup_{P \in \mathbf{P}_{t_i} \cup \mathbf{P}_{t_j}} P$, for any choice of $\mathbf{P}_{t_i}, \mathbf{P}_{t_j}$.*

Proof. Suppose, by way of contradiction, that there exists an arc $a \subset C \setminus \cup_{P \in \mathbf{P}_{t_i} \cup \mathbf{P}_{t_j}} P$. Let $C = \widehat{C_1} \cup \widehat{C_2}$, where each $\widehat{C_i}$ is a $\mathbf{P}_{t_l}$-crossing. Without loss of generality, we assume $a \subset \widehat{C_1}$. Then, $\mathcal{N} \setminus a$ is a regular 3-pair network by choosing $\mathbf{P}_{t_l}$ such that $\widehat{C_2} \subset \cup_{P \in \mathbf{P}_{t_l}} P$, which violates the minimality of $\mathcal{N}$. $\square$

Lemma 5.2. *Any semi-cycle is of order 1.*

Proof. First of all, we choose $\mathbf{P}_{t_i}$, $\mathbf{P}_{t_j}$ such that $\mathcal{N}_{t_i,t_j}$ is a minimal $(3,2)$-network. Then, by Theorems 4.6 and 4.7, $\mathcal{N}_{t_i,t_j}$ is topologically equivalent to one of the eight networks as shown in Fig. 6 and Fig. 7. It can be readily verified that none of the three networks in Fig. 6 and the network in Fig. 7 (a) can cover any semi-cycle; and each of the other four networks in Fig. 7 can cover a semi-cycle of order 1 but cannot cover a semi-cycle of order 2, which completes the proof. $\square$

Lemma 5.3. *For any $\mathbf{P}_{t_i}$-semi-cycle C and any $\mathbf{P}_{t_j}$-semi-cycle C', $C \cap C' = \emptyset$.*

Proof. Suppose, by way of contradiction, there exists $v \in C \cap C'$. If v is the head of C, then by Lemma 5.1 and the regularity of $\mathcal{N}$, v has an unique incoming arc and two outgoing arcs such that one is a $\mathbf{P}_{t_j}$-arc and the other is a $\mathbf{P}_{t_l}$-arc, for any choice of $\mathbf{P}_{t_j}$ and $\mathbf{P}_{t_l}$. It is easy to see that $v \notin C'$ since any point in a $\mathbf{P}_{t_j}$-semi-cycle cannot have the above property. Thus, we conclude that v is not the head of C. Similarly, we know that v is not the tail of C. In the same way, v is not the head/tail of C'.

Suppose $C = \widehat{C_1} \cup \widehat{C_2}$ and $C' = \widehat{C_1'} \cup \widehat{C_2'}$. By 1) of Theorem 3.3 and the above discussions, $C \cap C'$ has no isolated points and hence we assume, without loss of generality, $[u, v] \subseteq \widehat{C_1} \cap \widehat{C_1'}$ is a connected component of $C \cap C'$. Clearly, we have $deg^+(u) \geq 2$ and $deg^-(v) \geq 2$. However, by the definitions of $\mathbf{P}_{t_i}$-semi-cycle and $\mathbf{P}_{t_j}$-semi-cycle, there exists a choice of $\mathbf{P}_{t_i}$ such that $[u, v] \subset \widehat{C_1} \nsubseteq \cup_{P \in \mathbf{P}_{t_i}} P$ and a choice of $\mathbf{P}_{t_j}$ such that $[u, v] \subset \widehat{C_1'} \nsubseteq \cup_{P \in \mathbf{P}_{t_j}} P$. Hence, $[u, v]$ is composed of pure $\mathbf{P}_{t_l}$-arcs, which, however, is impossible since $deg^+(u) \geq 2$ and $deg^-(v) \geq 2$ and the lemma is then proved. $\square$

Consider a $\mathbf{P}_{t_l}$-semi-cycle C which is covered by $\mathcal{N}_{t_i,t_j}$ for some choice of $\mathbf{P}_{t_i}$ and $\mathbf{P}_{t_j}$. For an arc $a \subset C$, if a is a $\mathbf{P}_{t_i}$-arc, by Lemma 5.3, a does not belong to any $\mathbf{P}_{t_i}$-semi-cycle, and then by Remark 3.7, a is a $\mathbf{P}_{t_i}$-arc for *all* choice of $\mathbf{P}_{t_i}$. Hence, in the following, we define the type of a $\mathbf{P}_{t_l}$-semi-cycle C according to how it is embedded into $\mathcal{N}_{t_i,t_j}$ for *all* choice of $\mathbf{P}_{t_i}$ and $\mathbf{P}_{t_j}$. More specifically, suppose $C = \widehat{C_1} \cup \widehat{C_2}$, where $\widehat{C_1}$ and $\widehat{C_2}$ are two paths with the same head and tail. Then, a $\mathbf{P}_{t_l}$-semi-cycle C must be of the following three types:

Type 1: There exist $P \in \mathbf{P}_{t_i}$ and $Q \in \mathbf{P}_{t_j}$ such that $\widehat{C_1} \subset P$ and $\widehat{C_2} \subset Q$. Fig. 7 (b) may cover a $\mathbf{P}_{t_3}$-semi-cycle C of this type such that the head of C is the tail of $r_j(1) = g_j(1)$ and the tail of C is $r_j(3) = g_j(3)$;

Type 2: None of $\widehat{C_1}$ and $\widehat{C_2}$ is covered by a path $P \in \mathbf{P}_{t_i}$ or a path $Q \in \mathbf{P}_{t_j}$. Fig. 7 (c) may cover a $\mathbf{P}_{t_3}$-semi-cycle C of this type such that the head of C is the tail of $r_j(1) = g_j(1)$ and the tail of C is $r_l(3) = g_i(3)$;

Type 3: Only one of $\widehat{C_1}$ and $\widehat{C_2}$ is covered by a path $P \in \mathbf{P}_{t_i}$ or a path $Q \in \mathbf{P}_{t_j}$. Fig. 7 (d) may cover a $\mathbf{P}_{t_3}$-semi-cycle C of this type such that the head of C is the tail of $r_j(1) = g_j(1)$ and the tail of C is $r_l(3) = g_j(2)$.

Remark 5.4. If $\mathcal{N}_{t_1,t_2}$ has the topology as in Fig. 7 (e), then it may cover the following $\mathbf{P}_{t_3}$-semi-cycles:

(1) The head of C is the tail of $r_j(1) = g_j(1)$ and the tail of C is the head of $r_j(3) = g_j(3)$. C is of type 1;

(2) The head of C' is the tail of $r_j(1) = g_j(1)$ and the tail of C' is the head of $r_l(3) = g_j(2)$. C' is of type 3;

(3) The head of C'' is the tail of $g_i(2) = r_j(2)$ and tail of C'' is the head of $r_j(3) = g_j(3)$. C'' is of type 3.

For a minimal 3-pair network $\mathcal{N}$, we say $\mathcal{N}$ *contains a* $\mathbf{P}_{t_l}$*-semi-cycle* C if there exists a choice of $\mathbf{P}_{t_l}$ such that C is a $\mathbf{P}_{t_l}$-semi-cycle. For example, in Remark 5.4, if both $\mathbf{P}_{t_3}$-semi-cycle C' and $\mathbf{P}_{t_3}$-semi-cycle C'' are contained in $\mathcal{N}$, then $\mathbf{P}_{t_3}$-semi-cycle C is also contained in $\mathcal{N}$ (Notice that $C \subseteq C' \cup C''$). That is to say, if there exists a choice of $\mathbf{P}_{t_3}$ such that C' is a $\mathbf{P}_{t_3}$-semi-cycle and a (possibly different) choice of $\mathbf{P}_{t_3}$ such that C'' is a $\mathbf{P}_{t_3}$-semi-cycle, then there exists a choice of $\mathbf{P}_{t_3}$ such that C is a $\mathbf{P}_{t_3}$-semi-cycle. However, it is easy to see that there does not exist a choice of $\mathbf{P}_{t_3}$ such that all of C, C' and C'' are $\mathbf{P}_{t_3}$-semi-cycles at once.

Lemma 5.5. *Suppose that $\mathcal{N}$ contains a $\mathbf{P}_{t_l}$-semi-cycle. If $\mathcal{N}$ contains no $\mathbf{P}_{t_l}$-semi-cycle of type* 1*, then it contains an* unique *$\mathbf{P}_{t_l}$-semi-cycle of type* 2 *or* 3*.*

Proof. Suppose $C \subset \mathcal{N}_{t_i,t_j}$ is a $\mathbf{P}_{t_l}$-semi-cycle for some choice of $\mathbf{P}_{t_l}$, where $\mathcal{N}_{t_i,t_j}$ is a minimal $(3,2)$-network. If $\mathcal{N}_{t_i,t_j}$ has the topology as in Fig. 7 (b) (*resp.* (c), (d)), then by Lemma 5.1, C is the unique $\mathbf{P}_{t_l}$-semi-cycle of type 1 (*resp.* 2, 3). If $\mathcal{N}_{t_i,t_j}$ has topology as in Fig. 7 (d), then, it follows from the discussions below Remark 5.4 and the fact that $\mathcal{N}$ contains no $\mathbf{P}_{t_l}$-semi-cycle of type 1 that it contains an unique $\mathbf{P}_{t_l}$-semi-cycle of type 3. □

We have the following theorem.

Theorem 5.6. *For all $l = 1, 2, 3$, if $\mathcal{N}$ contains no $\mathbf{P}_{t_l}$-semi-cycle of type* 1*, then there exists a choice of $\mathbf{P}$ such that all $\mathcal{N}_{t_1,t_2}$, $\mathcal{N}_{t_1,t_3}$ and $\mathcal{N}_{t_2,t_3}$ are minimal.*

Proof. If $\mathcal{N}$ is stable, then there is nothing to prove. Hence, we suppose in the following that there exists a $\mathbf{P}_{t_l}$-semi-cycle for some $l \in \{1, 2, 3\}$. If $\mathcal{N}$ contains no $\mathbf{P}_{t_l}$-semi-cycle of type 1, then by Lemma 5.5, $\mathcal{N}$ contains an unique $\mathbf{P}_{t_l}$-semi-cycle $C = \widehat{C_1} \cup \widehat{C_2}$ of type 2 or 3. By the definition of type 2/type 3 semi-cycle, we assume, without loss of generality, $\widehat{C_1}$ is neither covered by a path of $\mathbf{P}_{t_i}$ nor by a path of $\mathbf{P}_{t_j}$ for any choice of $\mathbf{P}_{t_i}$ and $\mathbf{P}_{t_j}$. Clearly, if we choose $\mathbf{P}_{t_l}$ such that $\widehat{C_2} \subset P$, $P \in \mathbf{P}_{t_l}$, then C is neither a $\mathbf{P}_{t_l}$-semi-cycle in $\mathcal{N}_{t_i,t_l}$ nor a $\mathbf{P}_{t_l}$-semi-cycle in $\mathcal{N}_{t_j,t_l}$, for any choice of $\mathbf{P}_{t_i}$ and $\mathbf{P}_{t_j}$.

Since $\mathcal{N}$ contains no $\mathbf{P}_{t_l}$-semi-cycle of type 1 for all $l = 1, 2, 3$, we can always choose $\mathbf{P}_{t_1}$, $\mathbf{P}_{t_2}$ and $\mathbf{P}_{t_3}$ as above such that there is no semi-cycle in $\mathcal{N}_{t_1,t_2}$ $\mathcal{N}_{t_1,t_3}$ and $\mathcal{N}_{t_2,t_3}$. The theorem then follows from Lemma 4.1. □

Corollary 5.7. *For all $l = 1, 2, 3$, if $\mathcal{N}$ contains no $\mathbf{P}_{t_l}$-semi-cycle C of type* 1 *such that C is either an uppermost $\mathbf{P}_{t_l}$-semi-cycle in $\mathcal{N}_{t_i,t_l}$ or an uppermost $\mathbf{P}_{t_l}$-semi-cycle in $\mathcal{N}_{t_j,t_l}$ then there exists a choice of $\mathbf{P}$ such that all $\mathcal{N}_{t_1,t_2}$, $\mathcal{N}_{t_1,t_3}$ and $\mathcal{N}_{t_2,t_3}$ are pseudo-minimal.*

Proof. By definition, we suppose $C = \widehat{C_1} \cup \widehat{C_2}$ is a $\mathbf{P}_{t_l}$-semi-cycle of type 1 such that $\widehat{C_1}$ is covered by a path of $\mathbf{P}_{t_i}$ and $\widehat{C_2}$ is covered by a path of $\mathbf{P}_{t_j}$. Hence, if we choose $\mathbf{P}_{t_l}$ such that $\widehat{C_1} \subset P$, $P \in \mathbf{P}_{t_l}$, then C is a $\mathbf{P}_{t_l}$-semi-cycle in $\mathcal{N}_{t_j,t_l}$ but not a $\mathbf{P}_{t_l}$-semi-cycle in $\mathcal{N}_{t_i,t_l}$; and if we choose $\mathbf{P}_{t_l}$ such that $\widehat{C_2} \subset P$, $P \in \mathbf{P}_{t_l}$, then C is a $\mathbf{P}_{t_l}$-semi-cycle in $\mathcal{N}_{t_i,t_l}$ but not a $\mathbf{P}_{t_l}$-semi-cycle in $\mathcal{N}_{t_j,t_l}$ (see Fig. 8 (b) for an illustrative example). Hence, according to Theorem 5.6 and by the assumption, we can choose $\mathbf{P}_{t_l}$ such that there is no *uppermost* $\mathbf{P}_{t_l}$-semi-cycle in $\mathcal{N}_{t_i,t_l}$ and $\mathcal{N}_{t_j,t_l}$. For all $l = 1, 2, 3$, by choosing $\mathbf{P}_{t_l}$ as above, we obtain a choice of $\mathbf{P}$ such that there is no uppermost semi-cycle in each of $\mathcal{N}_{t_1,t_2}$, $\mathcal{N}_{t_1,t_3}$ and $\mathcal{N}_{t_2,t_3}$, which completes the proof. □

In the next section, we will prove that if all $\mathcal{N}_{t_1,t_2}$, $\mathcal{N}_{t_1,t_3}$ and $\mathcal{N}_{t_2,t_3}$ are pseudo-minimal, then there is a feasible linear multi-flow with rate $(1, 1, 1)$. To that end, we need first give a detailed classification of such networks and give a forbidden structure in the next subsection.

5.2 A Forbidden Structure

Given a choice of $\mathbf{P}$ of a minimal 3-pair network $\mathcal{N}$, for each $i \neq j$, consider the regular $(3,2)$-network $\mathcal{N}_{t_i,t_j}$ and let

$$m^i_{i,j} := \{l' : \ell(P_{s_{l'},t_i}) = 1\},$$

where $\ell(P_{s_{l'},t_i})$ ($P_{s_{l'},t_i} \in \mathbf{P}_{t_i}$) denotes the number of connected components of $P_{s_{l'},t_i}|_{\mathbf{P}_{t_j}}$ (see (5) for the definition). Note that we neglect the order of the two subscripts of m, i.e., $m^i_{i,j} = m^i_{j,i}$ and for notational convenience, if $m^i_{i,j} = \{l\}$, we may simply write $m^i_{i,j} = l$.

In Fig. 8 (a), one can check that $m^1_{1,2} = 1$, $m^2_{1,2} = 3$, $m^1_{1,3} = 1$, $m^3_{1,3} = 3$, $m^2_{2,3} = 1$ and $m^3_{2,3} = 3$; in Fig. 8 (b), if we choose P_{s_2,t_3} as $[s_2, v_1, v_2, v_3, v_4, v_5, t_3]$, then $m^1_{1,2} = 1$, $m^2_{1,2} = 3$, $m^1_{1,3} = \{1,3\}$, $m^3_{1,3} = \{2,3\}$, $m^2_{2,3} = 3$ and $m^3_{2,3} = 1$; if we choose P_{s_2,t_3} as $[s_2, v_1, u_2, u_3, v_4, v_5, t_3]$, then $m^1_{1,2} = 1$, $m^2_{1,2} = 3$, $m^1_{1,3} = 1$, $m^3_{1,3} = 3$, $m^2_{2,3} = \{2,3\}$ and $m^3_{2,3} = \{1,2\}$. Hence, for both (a) and (b) of Fig. 8, and each choice of $\mathbf{P}$, we have $\mathcal{N}_{t_1,t_2}$, $\mathcal{N}_{t_1,t_3}$ and $\mathcal{N}_{t_2,t_3}$ are pseudo-minimal. The following theorem shows that some topological structure, that is characterized by $m^i_{i,j}$, will not occur in a minimal 3-pair network $\mathcal{N}$.

Theorem 5.8. *There does not exists a choice of* $\mathbf{P}$ *such that* $m^i_{i,j} = l$, $m^j_{i,j} = i$, $m^i_{i,l} = l$, $m^l_{i,l} = j$, $m^j_{j,l} = i$ *and* $m^l_{j,l} = j$ *all hold at once.*

Proof. Suppose, by way of contradiction, there exists a choice of $\mathbf{P}$ such that $m^1_{1,3} = m^1_{1,2} = 3$, $m^3_{1,3} = m^3_{2,3} = 2$, $m^2_{1,2} = m^2_{2,3} = 1$, where we have, without loss of generality, assume $i = 1$, $j = 2$ and $l = 3$.

We first consider the case that $\mathcal{N}$ is unstable, and without loss of generality, suppose there exists a $\mathbf{P}_{t_3}$-semi-cycle C. By Lemma 5.1, C is covered by $\mathcal{N}_{t_1,t_2}$, which is a minimal $(3,2)$-network such that $m^1_{1,2} = 3$ and $m^2_{1,2} = 1$. By inspecting (a)-(e) of Fig. 7, it is not hard to see that there does not exist a choice of $\mathbf{P}_{t_3}$, which can guarantee a $\mathbf{P}_{t_3}$-semi-cycle and $m^3_{1,3} = m^3_{2,3} = 2$, $m^1_{1,3} = 3$, $m^2_{3,2} = 1$ all at once, which completes the proof for this case.

In the following, we consider the case that $\mathcal{N}$ is stable. As before, for notational convenience, we will rewrite P_{s_i,t_1}, P_{s_i,t_2} and P_{s_i,t_3} as r_i, g_i and b_i, respectively and rewrite $\mathbf{P}_{t_1}$, $\mathbf{P}_{t_2}$ and $\mathbf{P}_{t_3}$ as r, g and b, respectively.

By applying 1) of Corollary 4.8 to $\mathcal{N}_{t_1,t_3}$ and $\mathcal{N}_{t_1,t_2}$, respectively, we obtain $\emptyset \neq b_1 \cap r_2 = b^r_1(2)$ and $\emptyset \neq g_2 \cap r_1 = g^r_2(2)$, respectively. Hence, we have $b_1 \cap g_2 \neq \emptyset$ (thus 3) of Corollary 4.8 can be applied to $\mathcal{N}_{t_2,t_3}$) and

$$\begin{cases} b_1 \cap g_2 \leq b_1 \cap r_2; \\ b_1 \cap g_2 \leq r_1 \cap g_2, \end{cases} \tag{9}$$

since otherwise $b^r_1(1,2)$ and $g^r_2(1,2)$ would form an r-crossing with order 2, contradicting the minimality of $\mathcal{N}$. Similarly, we have $g_3 \cap r_1 \neq \emptyset$, $r_2 \cap b_3 \neq \emptyset$ (thus 3) of Corollary 4.8 can be applied to $\mathcal{N}_{t_1,t_2}$ and $\mathcal{N}_{t_1,t_3}$), and the following inequalities:

$$\begin{cases} g_3 \cap r_1 \leq g_3 \cap b_1; \\ g_3 \cap r_1 \leq b_3 \cap r_1, \end{cases} \tag{10}$$

$$\begin{cases} r_2 \cap b_3 \leq g_2 \cap b_3; \\ r_2 \cap b_3 \leq r_2 \cap g_3, \end{cases} \tag{11}$$

According to 3) of Corollary 4.8, we have the following two cases:

Case 1) : There exist feasible $i \neq j$ such that $\mathcal{N}_{t_i,t_j}$ satisfy (6). In this case, we infer that either (i) $b_1 \cap g_3 \leq b_1 \cap g_2$; $r_1 \cap g_2 \leq r_1 \cap g_3$ or (ii) $r_2 \cap b_1 \leq r_2 \cap b_3$; $g_2 \cap b_3 \leq g_2 \cap b_1$ or (iii) $b_3 \cap r_1 \leq b_3 \cap r_2$; $g_3 \cap r_2 \leq g_3 \cap r_1$ holds. Without loss of generality, suppose (i) holds, then according to (9) and (10), we have

$$b_1 \cap g_3 \leq b_1 \cap g_2 \leq r_1 \cap g_2 \leq r_1 \cap g_3 \leq g_3 \cap b_1,$$

which implies the existence of a cycle and contradicts the acyclicity of $\mathcal{N}$.

Case 2) : None of $\mathcal{N}_{t_1,t_2}$, $\mathcal{N}_{t_1,t_3}$ and $\mathcal{N}_{t_2,t_3}$ satisfy (6). In this case, either (7) or (8) holds for each of $\mathcal{N}_{t_i,t_j}$. If one of above (i)-(iii) holds, then we can obtain a contradiction by the same arguments as in Case 1); otherwise, we have the following two subcases:

(2.1) $b_1 \cap g_3 \leq b_1 \cap g_2 \leq b_3 \cap g_2; g_3 \cap r_2 \leq g_3 \cap r_1 \leq g_2 \cap r_1; r_2 \cap b_1 \leq r_2 \cap b_3 \leq r_1 \cap b_3;$

(2.2) $b_3 \cap g_2 \leq b_1 \cap g_2 \leq b_1 \cap g_3; g_2 \cap r_1 \leq g_3 \cap r_1 \leq g_3 \cap r_2; r_1 \cap b_3 \leq r_2 \cap b_3 \leq r_2 \cap b_1.$

According to (9), (10) and (11), for Case (2.1), we have

$$b_1 \cap g_3 \leq b_1 \cap g_2 \leq b_1 \cap r_2 \leq b_3 \cap r_2 \leq g_3 \cap r_2 \leq g_3 \cap r_1 \leq g_3 \cap b_1;$$

and for Case (2.2), we have

$$b_3 \cap g_2 \leq b_1 \cap g_2 \leq r_1 \cap g_2 \leq r_1 \cap g_3 \leq r_1 \cap b_3 \leq r_2 \cap b_3 \leq g_2 \cap b_3.$$

Hence, both cases yield a cycle, contradicting the acyclicity of $\mathcal{N}$. The proof is then completed. $\square$

6 Main Result

In this section, we shall establish the Langberg-Médard multiple unicast conjecture for $k = 3$. Without specified otherwise, we assume $\mathcal{N}$ is a minimal 3-pair network with a choice of $\mathbf{P}$. We first introduce some basic tools developed in our previous work [1]-[5].

6.1 $\mathcal{S}_{\mathcal{N}}$ and $g_s(\mathcal{C})$

For a strongly reachable k-pair network $\mathcal{N} = (V, A, S, T)$, given a choice of $\mathbf{P}$, for any arc $a \in A$, define

$$s(a) := \{(i,j) \in [k] \times [k] : a \subset P_{s_i,t_j} \in \mathbf{P}\} \text{ and } \mathcal{S}_{\mathcal{N}} := \{s(a) : a \in A\},$$

where $[k] := \{1, 2, \ldots, k\}$.

The following seemingly trivial lemma is fundamental in our treatment.

Lemma 6.1. *If $P_{s_{i_1},t_{j_1}} \cap P_{s_{i_2},t_{j_2}} = \emptyset$, then $\{(i_1,j_1),(i_2,j_2)\} \nsubseteq s$ for any $s \in \mathcal{S}_{\mathcal{N}}$.*

Noticing that each arc of a strongly reachable network is passed by at most one path in $\mathbf{P}_{t_j} = \{P_{s_1,t_j}, P_{s_2,t_j}, \ldots, P_{s_k,t_j}\}$, we have

$$\mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S}_k := \{\{(i_1,j_1),(i_2,j_2),\ldots,(i_r,j_r)\} \subseteq [k] \times [k] : j_1, j_2, \ldots, j_r \text{ are all distinct}\}.$$

For a minimal 3-pair network $\mathcal{N}$, the following observation is useful.

Lemma 6.2. *If there exists h such that $h \in m_{i,j}^{l_1} \cap m_{i,l}^{l_2} \cap m_{j,l}^{l_3}$ for some feasible l_1, l_2, l_3 and distinct i, j, l, then either $\{(h,i),(h,j)\} \notin \mathcal{S}_{\mathcal{N}}$ or $\{(h,i),(h,l)\} \notin \mathcal{S}_{\mathcal{N}}$.*

Proof. Without loss of generality, we assume $h = 1$. Then, noticing the paths P_{s_1,t_1}, P_{s_1,t_2} and P_{s_1,t_3} have a same tail s_1, and for each feasible $i \neq j$, $P_{s_1,t_i} \cap P_{s_1,t_j}$ has an unique connected component, we see that at most one of $\{(1,1),(1,2)\}$, $\{(1,1),(1,3)\}$, $\{(1,2),(1,3)\}$ belongs to $\mathcal{S}_{\mathcal{N}}$, which completes the proof. $\square$

For a tuple of $k \times k$ matrices $\mathcal{C} = ((c_{i,j}^{(1)}),(c_{i,j}^{(2)}),\ldots,(c_{i,j}^{(k)}))$ satisfying (4), given $s \in \mathcal{S}_{\mathcal{N}}$ and $l \in [k]$, we define

$$g_s^{(l)}(\mathcal{C}) := \sum_{(i,j)\in s} c_{i,j}^{(l)} \quad \text{and} \quad g_s(\mathcal{C}) := \sum_{l=1}^{k} |g_s^{(l)}(\mathcal{C})|.$$

The following theorem, whose proof is straightforward and thus omitted, will be used as a key tool to establish our results.

Theorem 6.3. *$\mathcal{C}$ is a feasible multi-flow with rate $(1,1,\ldots,1)$ if $g_s(\mathcal{C}) \leq 1$ for any $s \in \mathcal{S}_{\mathcal{N}}$.*

To avoid cumbersome wording, henceforth, $\mathcal{N}$ is said to be *(linearly) solvable* if there exists a (linear) feasible multi-flow $\mathcal{F}$ with rate $(1,1,\ldots,1)$; meanwhile $\mathcal{F}$ is called a *(linear) routing solution* of $\mathcal{N}$.

6.2 Linear Routing Solutions

We first introduce some notations. For $s = \{(i_1,j_1),(i_2,j_2),\ldots,(i_{\alpha(s)},j_{\alpha(s)})\} \in \mathcal{S}_{\mathcal{N}}$, define the following multi-set:

$$Ind_s := \{i_1, j_1, i_2, j_2, \ldots, i_{\alpha(s)}, j_{\alpha(s)}\},$$

where $\alpha(s)$ denotes the size of s. For any feasible l, denote by $m_{Ind_s}(l)$ the multiplicity of l in Ind_s (if $l \notin Ind_s$, then $m_{Ind_s}(l) = 0$). An element $(i,j) \in s$ is said to be *diagonal* if $i = j$; otherwise *non-diagonal*. We use $\gamma(s)$ to denote the number of diagonal elements in s. For a quick example, consider $s = \{(1,1),(2,2),(1,3)\} \subseteq [3] \times [3]$. Then, $Ind_s = \{1,1,2,2,1,3\}$, $m_{Ind_s}(1) = 3$, $m_{Ind_s}(2) = 2$, $m_{Ind_s}(3) = 1$, $\alpha(s) = 3$ and $\gamma(s) = 2$.

Lemma 6.4. *Let*

$$\mathcal{C} = \left(\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{-1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{-1}{4} \end{pmatrix}, \begin{pmatrix} \frac{-1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{-1}{4} \end{pmatrix}, \begin{pmatrix} \frac{-1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{-1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \right).$$

Then, for any $s \in \mathcal{S}_k$, $g_s(\mathcal{C}) > 1$ if and only if $\alpha(s) = 3$ and $\gamma(s) = 0$.

Proof. The result can be obtained by considering the following 4 cases.

1) $\gamma(s) = 0$. In this case, it can be verified that

$$g_s(\mathcal{C}) = \frac{1}{4}\sum_{i=1}^{3} m_{Ind_s}(i) = 2\alpha(s) = \begin{cases} \frac{1}{2}, & \alpha(s) = 1; \\ 1, & \alpha(s) = 2; \\ \frac{3}{2}, & \alpha(s) = 3. \end{cases}$$

2) $\gamma(s) = 1$. In this case, it can be verified that

$$g_s(\mathcal{C}) = \begin{cases} \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1, & \alpha(s) = 1; \\ \frac{3}{4} + \frac{1}{4} = 1, & s = \{(i,i),(i,j)\}, \text{ where } i \neq j; \\ \frac{1}{2}, & s = \{(i,i),(k,j)\}, \text{ where } i,j,k \text{ are distinct;} \\ 1, & \alpha(s) = 3. \end{cases}$$

3) $\gamma(s) = 2$. In this case, it can be verified that $g_s(\mathcal{C}) = 1$.

4) $\gamma(s) = 3$. In this case, it obviously holds true that $g_s(\mathcal{C}) = 0$.

□

The following lemma can be obtained via straightforward computations and thereby we omit its proof.

Lemma 6.5. *If*

$$\begin{aligned} \mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S} := \{&\{(i,j)\} : 1 \leq i,j \leq 3\} \\ &\cup \{\{(i,1),(j,2)\} : i = 2,3; j = 1,2,3\} \cup \{\{(1,1),(1,2)\}\} \\ &\cup \{\{(i,1),(l,3)\} : i = 1,2; l = 1,2,3\} \cup \{\{(3,1),(3,3)\}\} \\ &\cup \{\{(j,2),(l,3)\} : j,l = 1,2,3\} \\ &\cup \{\{(1,1),(1,2),(l,3)\} : l = 1,2,3\} \cup \{\{(2,1),(j,2),(l,3)\} : j,l = 1,2,3\} \\ &\cup \{\{(3,1),(j,2),(3,3)\} : j = 1,2,3\}. \end{aligned}$$

Then

$$\mathcal{C} = \left(\begin{pmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{-1}{4} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{-1}{4} \end{pmatrix}, \begin{pmatrix} \frac{-1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{-1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \right)$$

is a linear routing solution of $\mathcal{N}$.

We are now ready for the following theorem.

Theorem 6.6. *If there exist distinct* $i,j,l \in \{1,2,3\}$ *such that* $m^i_{i,j} \cap \{i,j\} \neq \emptyset$ *and* $m^i_{i,l} \cap \{i,l\} \neq \emptyset$, *then* $\mathcal{N}$ *is linearly solvable.*

Proof. Suppose there exist distinct $i,j,l \in \{1,2,3\}$ such that $m^i_{i,j} \cap \{i,j\} \neq \emptyset$ and $m^i_{i,l} \cap \{i,l\} \neq \emptyset$. To prove the theorem, we need to consider the following three cases:

1) $i \in m^i_{i,j} \cap m^i_{i,l}$;

2) $j \in m^i_{i,j}$ and $l \in m^i_{i,l}$;

3) $i \in m^i_{i,j}$ and $l \in m^i_{i,l}$.

In the remainder of the proof, without loss of generality, we assume $i = 1$, $j = 2$ and $l = 3$.

For Case 1), since $1 \in m^1_{1,2} \cap m^1_{1,3}$, then P_{s_1,t_1} is disjoint from the strongly reachable 2-pair network $\mathcal{N}'$ deduced by paths $P_{s_2,t_2}, P_{s_3,t_2}, P_{s_2,t_3}, P_{s_3,t_3}$. According to [3], $\mathcal{N}'$ always has a linear routing solution

$$\left(\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{-1}{4} \end{pmatrix}, \begin{pmatrix} \frac{-1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right).$$

Hence, $\mathcal{N}$ has the following linear routing solution:

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{4} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix} \right).$$

For Case 2), consider all $s \in \mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S}_3$ such that $\alpha(s) = 3$. Let $s = \{(l_1, 1), (l_2, 2), (l_3, 3)\}$. If $l_1 = 1$, then obviously $\gamma(s) \neq 0$; if $l_1 = 2$, then since $2 \in m^1_{1,2}$, we have $l_2 = 2$ and hence $\gamma(s) \neq 0$; and if $l_1 = 3$, since $3 \in m^1_{1,3}$, we have $l_3 = 3$ and hence $\gamma(s) \neq 0$. Thus, for any $s \in \mathcal{S}_{\mathcal{N}}$ such that $\alpha(s) = 3$, we have $\gamma(s) \neq 0$. By Lemma 6.4,

$$\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{-1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{-1}{4} \end{pmatrix}, \begin{pmatrix} \frac{-1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{-1}{4} \end{pmatrix}, \begin{pmatrix} \frac{-1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{-1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \right).$$

is a linear routing solution of $\mathcal{N}$.

For Case 3), since $1 \in m^1_{1,2}$ and $3 \in m^1_{1,3}$, we deduce from Lemma 6.1 that $\mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S}$, where $\mathcal{S}$ is defined in Lemma 6.5. It then immediately follows that $\mathcal{N}$ is solvable, as desired. $\square$

We also need the following lemmas, which can be established by straightforward computations and whose proofs are then omitted.

Lemma 6.7. *Let*

$$\begin{aligned} \mathcal{S} := \{&\{(i,j)\} : 1 \leq i, j \leq 3\} \\ &\cup \{\{(i,1),(j,2)\} : i = 1,2; j = 2,3\} \cup \{\{(1,1),(1,2)\},\{(3,1),(3,2)\}\} \\ &\cup \{\{(i,1),(l,3)\} : i = 2,3; l = 1,3\} \cup \{\{(1,1),(1,3)\},\{(2,1),(2,3)\}\} \\ &\cup \{\{(j,2),(l,3)\} : j = 2,3; l = 1,2\} \cup \{\{(1,2),(1,3)\},\{(3,2),(3,3)\}\} \\ &\cup \{\{(1,1),(j,2),(1,3)\} : j = 1,2,3\} \cup \{\{(2,1),(2,2),(l,3)\} : l = 1,2\} \\ &\cup \{\{(2,1),(3,2),(l,3)\} : l = 1,2,3\} \cup \{\{(3,1),(3,2),(l,3)\} : l = 1,3\}. \end{aligned}$$

If $\mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S} \setminus \{(1,1),(1,2)\}$, then

$$\left(\begin{pmatrix} \frac{8}{14} & \frac{7}{14} & \frac{-1}{14} \\ \frac{3}{14} & \frac{-5}{14} & \frac{2}{14} \\ \frac{3}{14} & \frac{-2}{14} & \frac{-1}{14} \end{pmatrix}, \begin{pmatrix} \frac{-3}{14} & \frac{7}{14} & \frac{-4}{14} \\ \frac{3}{14} & \frac{7}{14} & \frac{4}{14} \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{-3}{14} & 0 & \frac{3}{14} \\ 0 & \frac{-2}{14} & \frac{2}{14} \\ \frac{3}{14} & \frac{2}{14} & \frac{9}{14} \end{pmatrix} \right)$$

is a linear routing solution of $\mathcal{N}$. If $\mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S} \setminus \{\{(1,1),(j,2),(1,3)\} : j = 2,3\}$, then

$$\left(\begin{pmatrix} \frac{6}{12} & \frac{3}{12} & \frac{3}{12} \\ \frac{3}{12} & \frac{-3}{12} & 0 \\ \frac{3}{12} & 0 & \frac{-3}{12} \end{pmatrix}, \begin{pmatrix} \frac{-3}{12} & \frac{4}{12} & \frac{-1}{12} \\ \frac{3}{12} & \frac{7}{12} & \frac{2}{12} \\ 0 & \frac{1}{12} & \frac{-1}{12} \end{pmatrix}, \begin{pmatrix} \frac{-3}{12} & \frac{1}{12} & \frac{2}{12} \\ 0 & \frac{-2}{12} & \frac{2}{12} \\ \frac{3}{12} & \frac{1}{12} & \frac{8}{12} \end{pmatrix} \right)$$

is a linear routing solution of $\mathcal{N}$.

Lemma 6.8. *Let*

$$\begin{aligned}\mathcal{S} := \{&\{(i,j)\} : 1 \le i,j \le 3\}\\ &\cup \{\{(i,1),(j,2)\} : i = 1,2; j = 2,3\} \cup \{\{(1,1),(1,2)\},\{(3,1),(3,2)\}\}\\ &\cup \{\{(i,1),(l,3)\} : i = 2,3; l = 1,3\} \cup \{\{(1,1),(1,3)\},\{(2,1),(2,3)\}\}\\ &\cup \{\{(j,2),(l,3)\} : j = 2,3; l = 1,3\} \cup \{\{(1,2),(1,3)\},\{(2,2),(2,3)\}\}\\ &\cup \{\{(1,1),(j,2),(1,3)\} : j = 1,2,3\} \cup \{\{(2,1),(2,2),(l,3)\} : l = 1,2,3\}\\ &\cup \{\{(2,1),(3,2),(l,3)\} : l = 1,3\} \cup \{\{(3,1),(3,2),(l,3)\} : l = 1,3\}.\end{aligned}$$

If $\mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S} \setminus \{(1,1),(1,2)\}$, *then*

$$\left(\begin{pmatrix} \frac{4}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{2}{8} & \frac{-2}{8} & 0 \\ \frac{2}{8} & \frac{-1}{8} & \frac{-1}{8} \end{pmatrix}, \begin{pmatrix} \frac{-2}{8} & \frac{3}{8} & \frac{-1}{8} \\ \frac{2}{8} & \frac{4}{8} & \frac{2}{8} \\ 0 & \frac{1}{8} & \frac{-1}{8} \end{pmatrix}, \begin{pmatrix} \frac{-2}{8} & 0 & \frac{2}{8} \\ 0 & \frac{-2}{8} & \frac{2}{8} \\ \frac{2}{8} & \frac{2}{8} & \frac{4}{8} \end{pmatrix}\right)$$

is a linear routing solution of $\mathcal{N}$. *If* $\mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S} \setminus \{\{(1,1),(1,3)\}\} \setminus \{\{(1,1),(j,2),(1,3)\} : j = 2,3\}$, *then*

$$\left(\begin{pmatrix} \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{-1}{6} & 0 \\ \frac{1}{6} & 0 & \frac{-1}{6} \end{pmatrix}, \begin{pmatrix} \frac{-2}{6} & \frac{3}{6} & \frac{-1}{6} \\ \frac{2}{6} & \frac{2}{6} & \frac{2}{6} \\ 0 & \frac{1}{6} & \frac{-1}{6} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ \frac{-1}{6} & \frac{-1}{6} & \frac{2}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{4}{6} \end{pmatrix}\right)$$

is a linear routing solution of $\mathcal{N}$.

6.3 Proof of Main Result

Lemma 6.9. *If all* $\mathcal{N}_{t_1,t_2}$, $\mathcal{N}_{t_1,t_3}$ *and* $\mathcal{N}_{t_2,t_3}$ *are pseudo-minimal, then* $\mathcal{N}$ *is linearly solvable.*

Proof. If there exist distinct $i,j,l \in \{1,2,3\}$ such that $m^i_{i,j} \cap \{i,j\} \neq \emptyset$ and $m^i_{i,l} \cap \{i,l\} \neq \emptyset$, then $\mathcal{N}$ is linearly solvable by Theorem 6.6. So we suppose in the remainder of the proof that for any distinct $i,j,l \in \{1,2,3\}$, either $m^i_{i,j} = l$ or $m^i_{i,l} = j$.

Without loss of generality, we assume $m^i_{i,j} = l$. Then, by the pseudo-minimality of $\mathcal{N}_{t_i,t_j}$, we have $m^j_{i,j} \cap \{i,j\} \neq \emptyset$. So, we assume $m^j_{l,j} = i$ since otherwise $\mathcal{N}$ is linearly solvable according to Theorem 6.6. Then, by the pseudo-minimality of $\mathcal{N}_{t_j,t_l}$, we have $m^l_{l,j} \cap \{l,j\} \neq \emptyset$. So, we assume $m^l_{l,i} = j$ since otherwise $\mathcal{N}$ is linearly solvable according to Theorem 6.6. Then, by the pseudo-minimality of $\mathcal{N}_{t_i,t_l}$, we have $m^i_{l,i} \cap \{i,l\} \neq \emptyset$.

By the above discussions, we finally have $l \in m^i_{i,j}$, $j \in m^l_{i,l}$ and $i \in m^j_{j,l}$. Consider the following cases: 1) $j \in m^j_{i,j}$, $i \in m^i_{i,l}$ and $l \in m^l_{j,l}$; 2) $i \in m^j_{i,j}$, $i \in m^i_{i,l}$ and $l \in m^l_{j,l}$; 2$'$) $j \in m^j_{i,j}$, $i \in m^i_{i,l}$ and $j \in m^l_{j,l}$; 2$''$) $j \in m^j_{i,j}$, $l \in m^i_{i,l}$ and $l \in m^l_{j,l}$; 3) $i \in m^j_{i,j}$, $i \in m^i_{i,l}$ and $j \in m^l_{j,l}$; 3$'$) $i \in m^j_{i,j}$, $l \in m^i_{i,l}$ and $l \in m^l_{j,l}$; 3$''$) $j \in m^j_{i,j}$, $l \in m^i_{i,l}$ and $j \in m^l_{j,l}$; 4) $i \in m^j_{i,j}$, $l \in m^i_{i,l}$ and $j \in m^l_{j,l}$. It can be readily verified that Cases 2$'$) and 2$''$) can be obtained from Case 2) with the relabelling $i \mapsto j$, $j \mapsto l$, $l \mapsto i$, and moreover, Cases 3$'$) and 3$''$) can be obtained from Case 3) with the relabelling $i \mapsto l$, $j \mapsto i$, $l \mapsto j$. So, in the following, we only need to consider Cases 1), 2), 3), 4).

For Case 1), without loss of generality, we assume $i = 1$, $j = 2$ and $l = 3$ and thus $2 \in m^2_{1,2}$, $1 \in m^1_{1,3}$ and $3 \in m^2_{2,3}$. Hence, P_{s_1,t_1}, P_{s_2,t_2} and P_{s_3,t_3} are pairwise disjoint and the network has a linear routing solution

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right).$$

For Case 2), without loss of generality, we assume $i = 1$, $j = 2$ and $l = 3$ and thus $1 \in m^2_{1,2} \cap m^1_{1,3} \cap m^2_{2,3}$; $2 \in m^3_{1,3}$ and $3 \in m^1_{1,2} \cap m^3_{2,3}$. It then follows from Lemmas 6.1 and 6.2 that either $\mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S} \setminus \{(1,1),(1,2)\}$ or $\mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S} \setminus \{\{(1,1),(j,2),(1,3)\} : j = 2,3\}$, where $\mathcal{S}$ is defined in Lemma 6.7. Hence $\mathcal{N}$ is solvable, which proves the theorem for Case 2).

For Case 3), without loss of generality, we assume $i = 1$, $j = 2$ and $l = 3$. It can be readily verified that $1 \in m^2_{1,2} \cap m^1_{1,3} \cap m^2_{2,3}$; $2 \in m^3_{1,3} \cap m^3_{2,3}$ and $3 \in m^1_{1,2}$. It then follows from Lemmas 6.1 and 6.2 that either $\mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S} \setminus \{(1,1),(1,2)\}$ or $\mathcal{S}_{\mathcal{N}} \subseteq \mathcal{S} \setminus \{\{(1,1),(1,3)\}\} \setminus \{\{(1,1),(j,2),(1,3)\} : j = 2,3\}$, where $\mathcal{S}$ is defined in Lemma 6.8. Hence $\mathcal{N}$ is solvable, which proves the theorem for Case 3).

For Case 4), if there exist feasible $i \neq j$ such that $m^i_{i,j}$ is not singleton, then $\mathcal{N}$ falls into at least one of the previous cases and hence is linearly solvable. So, we assume $m^i_{i,j}$ is singleton for all feasible $i \neq j$, which however is impossible by Theorem 5.8. The proof is then completed. □

We are now ready for our main result. As before, we will adopt the same notations as in the proof of Theorem 5.8, i.e., we will rewrite P_{s_i,t_1}, P_{s_i,t_2} and P_{s_i,t_3} as r_i, g_i and b_i, respectively and rewrite $\mathbf{P}_{t_1}$, $\mathbf{P}_{t_2}$ and $\mathbf{P}_{t_3}$ as r, g and b, respectively.

Theorem 6.10. *Any strongly reachable* 3*-pair network is linearly solvable.*

Proof. It suffices to prove that any minimal 3-pair network $\mathcal{N}$ is linearly solvable. By Corollary 5.7 and Theorem 6.9, we suppose $\mathcal{N}$ contain a b-semi-cycle C of type 1 such that C is either an uppermost b-semi-cycle in $\mathcal{N}_{t_1,t_3}$ or an uppermost b-semi-cycle in $\mathcal{N}_{t_2,t_3}$. Note that C is covered by a minimal $(3,2)$-network $\mathcal{N}_{t_1,t_2}$, we have the following two cases:

Case 1) : $\mathcal{N}_{t_1,t_2}$ has the topology as in Fig. 7 (b). In this case, as shown in Fig. 9 (a), let $r^g_i(1) = [s_i, v_1]$, $r^g_j(1) = [s_j, v_2]$, $r^g_l(1) = [s_l, v_3]$, $r^g_j(2) = [v_4, v_5]$, $g^r_j(2) = [u_4, u_5]$ and $r^g_j(3) = [v_6, v_7]$. Then, $C = \widehat{C_1} \cup \widehat{C_2}$, where $\widehat{C_1} = [v_2, v_4, v_5, v_6]$ and $\widehat{C_2} = [v_2, u_4, u_5, v_6]$. Since C is an uppermost b-semi-cycle in either $\mathcal{N}_{t_1,t_3}$ or $\mathcal{N}_{t_2,t_3}$, we have either $[s_j, v_2, v_4, v_5, v_6] \subset b_j$ or $[s_j, v_2, u_4, u_5, v_6] \subset b_j$. As a result, we have $s([s_j, v_2]) = \{(j,1),(j,2),(j,3)\}$ and by the regularity of $\mathcal{N}$, $a := b_j \cap [v_6, v_7]$ is an arc such that $s(a) = \{(j,1),(j,2),(j,3)\}$ as shown in Fig. 9 (b). For the two choices of b_j, we have:

$$\begin{aligned} &(1) \text{ if } \widehat{C_1} \subset b_j, \text{ then } s([v_4, v_5]) = \{(j,1),(i,2),(j,3)\}, s([u_4, u_5]) = \{(j,1),(l,2)\}; \\ &(2) \text{ if } \widehat{C_2} \subset b_j, \text{ then } s([u_4, u_5]) = \{(l,1),(j,2),(j,3)\}, s([v_4, v_5]) = \{(j,1),(i,2)\}. \end{aligned} \tag{12}$$

Now consider paths b_i and b_l. Clearly, $b_i \cap [s_j, v_2] = b_i \cap [v_4, v_5] = b_i \cap [u_4, u_5] = b_l \cap [s_j, v_2] = b_l \cap [v_4, v_5] = b_l \cap [u_4, u_5] = \emptyset$ and it is also not hard to see that $b_i \cap [v_6, v_7] = b_l \cap [v_6, v_7] = \emptyset$. We have the following two cases.

Case 1.1) $b_i \cap [s_l, v_3] = b_l \cap [s_i, v_1] = \emptyset$. For this case, we shall prove $\mathcal{N}$ is linearly solvable by Lemma 6.4. Noticing that $s([s_i, v_1] \cap b_i) = \{(i,1),(i,2),(i,3)\}$, $s([s_j, v_2]) = s(a) = \{(j,1),(j,2),(j,3)\}$ and $s([s_l, v_3] \cap b_l) = \{(l,1),(l,2),(l,3)\}$, and by (12), we choose

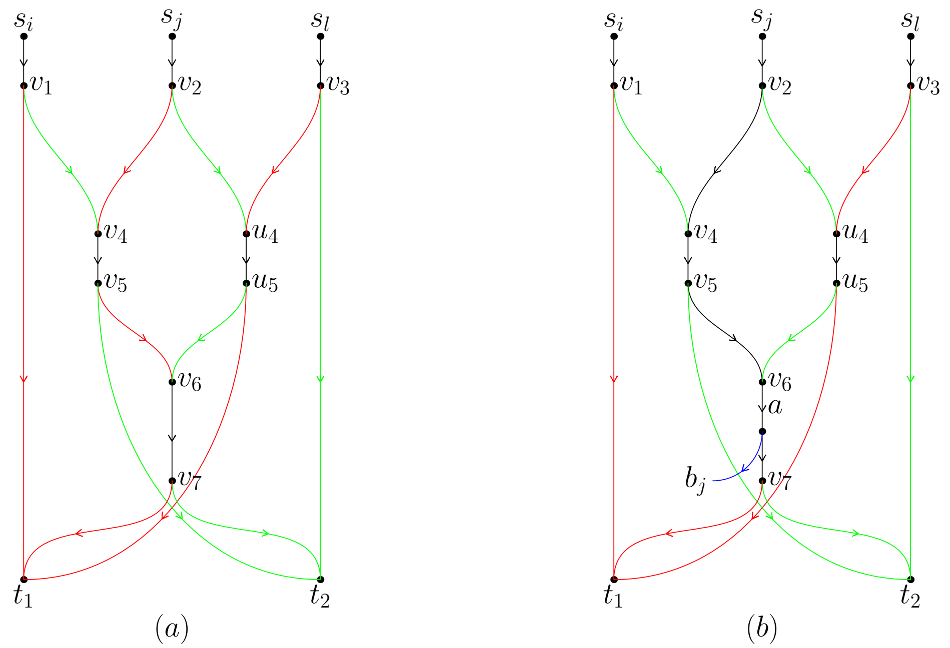

Figure 9: Illustrations for the proof of Case 1) of Theorem 6.10. In (b), we choose b_j such that $\widehat{C_1} \subset b_j$ and $s(b_j \cap [v_6, v_7]) = \{(j,1),(j,2),(j,3)\}$.

$\widehat{C_1} \subset b_j$ if $j = 1$ and choose $\widehat{C_2} \subset b_j$ if $j = 2$ (we arbitrarily choose $\widehat{C_1} \subset b_j$ or $\widehat{C_2} \subset b_j$ if $j = 3$). Then, for all $s \in \mathcal{S}_{\mathcal{N}}$ such that $\alpha(s) = 3$, we have $\gamma(s) \neq 0$, which completes the proof for this case.

Case 1.2) $b_i \cap [s_l, v_3] \neq \emptyset$ or $b_l \cap [s_i, v_1] \neq \emptyset$. Without loss of generality, we suppose $b_i \cap [s_l, v_3] \neq \emptyset$. In this case, $a' := b_i \cap [s_l, v_3]$ is an arc such that $s(a') = \{(l,1),(l,2),(i,3)\}$ and by the minimality of $\mathcal{N}$, it is not hard to see $b_l|_r = b_l|_g = b_l^r(1) = b_l^g(1)$. As a result, if $l = 3$, the network can be solved by

$$\left(\begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right),$$

otherwise, we have either $i = 3$ or $j = 3$. If $i = 3, j = 1$, we choose $\widehat{C_1} \subset b_j$; if $i = 3, j = 2$, we choose $\widehat{C_2} \subset b_j$; and if $j = 3$, we arbitrarily choose $\widehat{C_1} \subset b_j$ or $\widehat{C_2} \subset b_j$. For all the above choices of b_j, we have that for all $s \in \mathcal{S}_{\mathcal{N}}$ such that $\alpha(s) = 3$, $\gamma(s) \neq 0$, and hence $\mathcal{N}$ is linearly solvable by Lemma 6.4.

Case 2) : $\mathcal{N}_{t_1,t_2}$ has the topology as in Fig. 7 (e). In this case, we define v_1-v_7, u_4, u_5 in the same way as in Case 1) and let $r_l^g(2) = [w_4, w_5]$, as shown in Fig. 10 (a). Then, $\mathcal{N}$ contains the type 1 b-semi-cycle $C = \widehat{C_1} \cup \widehat{C_2}$, where $\widehat{C_1} = [v_2, v_4, v_5, v_6]$ and $\widehat{C_2} = [v_2, u_4, u_5, v_6]$. Let $(v_5, w_4, w_5, u_4) := [v_5, w_4, w_5, u_4] \setminus \{v_5, u_4\}$, we have two subcases according to Remark 5.4:

2.1) $(v_5, w_4, w_5, u_4) \cap (\cup_{l'=1}^3 b_{l'}) = \emptyset$. In this case, noticing that $\alpha(s([w_4, w_5])) = 2$ for any choices of b_j, similarly as in Case 1), $\mathcal{N}$ is linearly solvable.

2.2) $(v_5, w_4, w_5, u_4) \cap (\cup_{l'=1}^3 b_{l'}) \neq \emptyset$. In this case, since C is an uppermost b-semi-cycle in either $\mathcal{N}_{t_1,t_3}$ or $\mathcal{N}_{t_2,t_3}$, we have either $[s_j, v_2, v_4, v_5, v_6] \subset b_j$ or $[s_j, v_2, u_4, u_5, v_6] \subset b_j$ and hence

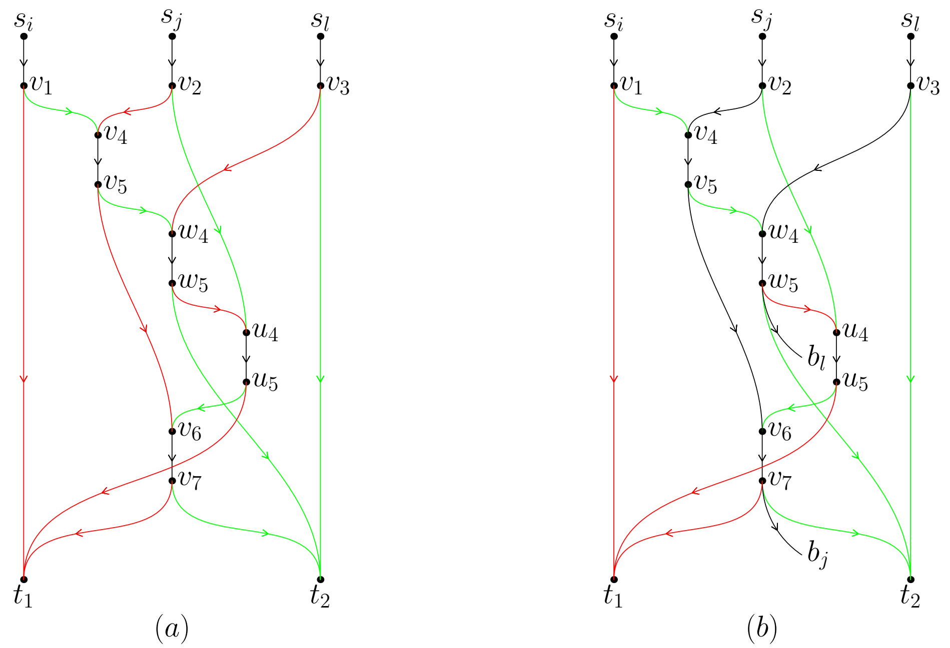

Figure 10: Illustrations for the proof of Case 2) of Theorem 6.10. In (b), we choose path b_j such that $\widehat{C_1} \subset b_j$ and $s(b_j \cap [v_6, v_7]) = \{(j,1),(j,2),(j,3)\}$. We have $[s_l, v_3, w_4] \subset b_l$ and hence $s(b_l \cap [w_4, w_5]) = \{(l,1),(i,2),(l,3)\}$.

$(v_5, w_4, w_5, u_4) \cap b_j = \emptyset$. Moreover, by the minimality of $\mathcal{N}$, $(v_5, w_4, w_5, u_4) \cap b_i = \emptyset$. Hence, we have $(v_5, w_4, w_5, u_4) \cap b_l \neq \emptyset$. Furthermore, we can see that $[s_l, v_3, w_4] \subset b_l$, as shown in (b) of Fig. 10, which further indicates $g_l|_r = g_l|_b = g_l^r(1) = g_l^b(1)$. Hence, if $l = 2$, $\mathcal{N}$ can be solved by

$$\left(\begin{pmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & -\frac{1}{4} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \right).$$

If $l \neq 2$, using the fact that $s([w_4, w_5] \cap b_l) = \{(l,1),(i,2),(l,3)\}$ and Lemma 6.4, the same arguments as in Case 2.1) will yield that $\mathcal{N}$ is linearly solvable, completing the proof. □

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